Motivic classes of stacks in finite characteristic and applications to stacks of Higgs bundles and bundles with connections

Ruoxi Li

University of Pittsburgh

Introduction

Our primary goal is to compute the motivic classes of stacks of Higgs bundles and bundles with connections in finite characteristic which was earlier done in characteristic 0 by Fedorov and Soibelmans in [FSS].

Motivation

One way of looking at motivic class computations is as a generalization of counting the number of points in an algebraic variety over a finite field.

Counting points of varieties over $\mathbb{F}_q$

Let $|X| := |X(\mathbb{F}_q)|$ denote the number of rational points of an algebraic variety X over a finite field $\mathbb{F}_q$ with q elements.

- $|\mathbb{A}^n| = q^n$.
- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathrm{GL}(n)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$ considered as the space of n linearly independent columns.

Counting points of algebraic stacks

- $|\mathbb{P}^n|$ can also be computed by presenting $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1} - \{0\}$ by the action of $\mathrm{GL}(1)$.
- We can try to do similar computations even when the group action is not free and the quotient is no longer a variety.
- Algebraic stacks generalize varieties by allowing points to have nontrivial automorphisms.
- Examples include quotients X/G of a variety by a non-free algebraic group action, moduli of curves, moduli of vector bundles, etc.

If $\mathcal{Y}$ is an algebraic stack, we can try to generalize point counting for varieties by defining the volume (also known as the mass) of $\mathcal{Y}$ as a weighted sum over all its $\mathbb{F}_q$-rational points:

$$|\mathcal{Y}| := \sum_{y \in \mathcal{Y}} \frac{1}{|\operatorname{Aut}(y)|}.$$

This is not guaranteed to converge. However, all the stacks we consider are of finite type, for which it converges (the sum is finite).

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Motivic classes in finite characteristic

Motivic classes of schemes

For any field k, one defines the abelian group $\mathrm{Mot}_{sch}(k)$ as the group generated by isomorphism classes of schemes of finite type over k modulo the following relations:

1) $[X] = [Y] + [X - Y]$ where Y is a closed subscheme of X,
2) $[X] = [Y]$ if there is a radicial and surjective morphism $X \to Y$ of schemes over k.
3) $[X] = [\mathrm{PGL}(r)][Y]$ for every $\mathrm{PGL}(r)$-torsor $X \to Y$.

The class $[X]$ in $\mathrm{Mot}_{sch}(k)$ is called the motivic class of the scheme X. We can give a ring structure for $\mathrm{Mot}_{sch}(k)$ by $[X]\cdot[Y] = [X \times_k Y]$ with the unit element $[\operatorname{Spec} k] = 1$.

Relation between the motivic classes

Similarly, we can define $\mathrm{Mot}(k)$ for algebraic stacks of finite type such that the stablizers of points are affine. For a scheme X, in [Eke] we have

$$[X/\mathrm{GL}(n)] = [X]/[\mathrm{GL}(n)].$$

There is a natural group isomorphism in [Li]:

$$\mathrm{Mot}_{sch}(k)[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1} | i > 0] \cong \mathrm{Mot}(k)$$

Let $\mathrm{F}^m \mathrm{Mot}(k)$ be the subgroup generated by the classes of stacks of dimension $\leq -m$. This is a ring filtration and we define the completed ring $\overline{\mathrm{Mot}}(k)$ as the completion of $\mathrm{Mot}(k)$ with respect to this filtration.

Motivic zeta function

We can define the motivic zeta function for every reduced quasi-projective scheme X:

$$Z_X(t) = \sum_{n \geq 0} [Sym^n X] \cdot t^n \in 1 + t \cdot \mathrm{Mot}_{sch}(k)[[t]]$$

where $Sym^n X$ is the symmetric product X^n/S_n (with the convention that $Sym^0 X = \operatorname{Spec} k$).

If $k = \mathbb{F}_q$, then the image of this series under $\# : \mathrm{Mot}_{sch}(k) \to \mathbb{Z}$ coincides with the local zeta function.

Plethystic exponents

Now we can consider the ring of formal power series in two variables $\overline{\mathrm{Mot}}(k)[[z, w]]$. Let $\overline{\mathrm{Mot}}(k)[[z, w]]^+$ denote the ideal of power series with vanishing constant term and let $(1 + \overline{\mathrm{Mot}}(k)[[z, w]]^+)^\times$ be the multiplicative group of series with constant term equal 1. We can define the plethystic exponent $\operatorname{Exp} : \overline{\mathrm{Mot}}(k)[[z, w]]^+ \to (1 + \overline{\mathrm{Mot}}(k)[[z, w]]^+)^\times$ by

$$\operatorname{Exp}\left(\sum_{r,d} A_{r,d} w^r z^d\right) = \prod_{r,d} Z_{A_{r,d}}(w^r z^d).$$

Hua's formula for bundles with automorphisms

In representation theory, we have Hua's formula for counting representations of quivers over finite fields in [Hua]. The formula also exists for counting Higgs bundles over finite fields as Theorem 4.9 in [MS].

Hua's formula

Let $\mathcal{A}ut^+$ denote the stack of HN-nonnegative bundles with automorphism and $\mathcal{A}ut^{ind,+} \subseteq |\mathcal{A}ut^+|$ is the constructible subset, where the underlying vector bundle is geometrically indecomposable.

The following theorem in [Li] is even new when $\mathrm{char}(k) = 0$:

$$1 + \sum_{r,d} [\mathcal{A}ut^+_{r,d}] w^r z^d = \operatorname{Exp}\left(\sum_{r,d} [\mathcal{A}ut^{ind,+}_{r,d}] w^r z^d\right).$$

Sketch of the proof I

Let $\mathcal{X}$ be a stack of finite type over a field k. Take $n > 0$ and $i \in [1, n]$, we have the i-th diagonal morphism $\mathcal{X}^{n-1} \to \mathcal{X}^n$. Let $\mathcal{Y}_i$ be the image for the composition $\mathcal{X}^{n-1} \to \mathcal{X}^n \to \mathcal{X}^n/S_n$; this is a constructible subset.

(1) Let $\wedge^n \mathcal{X} := |\mathcal{X}^n/S_n| - \cup_i \mathcal{Y}_i$.

(2) We define "virtual symmetric powers of stacks" by

$$S^n \mathcal{X} := \bigsqcup_{\mu, |\mu| = n} \prod_i \wedge^{a_i} \mathcal{X}.$$

Here $\mu = 1^{a_1} \ldots i^{a_i} \ldots$ is a partition of n.

For a stack $\mathcal{X}$, let $I\mathcal{X}$ denote its inertia stack, in [Li] we have the following theorem

$$\sum_{n \geq 0} [IS^n \mathcal{X}] t^n = \operatorname{Exp}(t[I\mathcal{X}]).$$

Sketch of the proof II

We define $\mathcal{A}ut^{n-ind,+} = I\mathcal{B}un^{n-ind,+}$ as a constructible subset of $\mathcal{A}ut^+$ consisting of (E, Φ), where geometrically $E \cong \bigoplus_{i=1}^n E_i$, E_i are indecomposable. Thus in [Li] we have the following proposition in $\mathrm{Mot}(k)[[z, w]]$:

$$[I\mathcal{B}un^{n-ind,+}] = [IS^n \mathcal{B}un^{ind,+}]$$

Sketch of the proof III

By the above Theorem and Proposition, now we have

$$\operatorname{Exp}\left([I\mathcal{B}un^{ind,+}]t\right) = \sum_{n \geq 0} [IS^n \mathcal{B}un^{ind,+}] t^n$$
$$= \sum_{n \geq 0} [I\mathcal{B}un^{n-ind,+}] t^n.$$

Plugging in $t = 1$, we obtain the required statement.

$$\operatorname{Exp}\left(\sum_{r,d} [\mathcal{A}ut^{ind,+}_{r,d}] w^r z^d\right) = 1 + \sum_{r,d} [\mathcal{A}ut^+_{r,d}] w^r z^d.$$

Applications to stacks of bundles with connections

Fix a smooth geometrically connected projective curve X over k.

Bundles with connections

A bundle with connection on X is a pair (E, ∇) where E is a vector bundle on X and $\nabla : E \to E \otimes \Omega_X$ is a k-linear morphism of sheaves satisfying Leibniz rule, i.e. for any open subset U of X, any $f \in H^0(U, \mathcal{O}_X)$ and any $s \in H^0(U, E)$ we have

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

Stacks of bundles with connections

Compared to the characteristic 0 case of [FSS], where every vector bundle with a connection has degree zero, in finite characteristic its degree is a multiple of the characteristic. We use $\mathcal{C}onn_{r,pd}$ to denote the moduli stack of rank r degree pd vector bundles with connections on X. In finite characteristic, the vector bundles with connections are not automatically semistable, so we restrict to the substack of semistable bundles with connections with the notation $\mathcal{C}onn^{ss}_{r,pd}$.

Compare the two motivic classes

Similarly we use $\mathcal{H}iggs^{ss}_{r,d}$ to denote the moduli stack of rank r degree d semistable Higgs bundles on X. For the characteristic 0 case in [FSS], we have

$$[\mathcal{C}onn_r(X)] = [\mathcal{H}iggs^{ss}_{r,0}(X)].$$

Formula for bundles with connections

The following formula in [Li] is very similar to the formula for $[\mathcal{H}iggs^{ss}_{r,d}(X)]$ in [FSS].

For finite characteristic p and sufficiently large e, we have $[\mathcal{C}onn^{ss}_{r,pd}] = C_{r,p(d+er)}$, where $C_{r,pd}$ is defined via the generating function

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} C_{r,pd} w^r z^{pd} = \operatorname{Exp}\left(\sum_{d/r=\tau} B_{r,pd} w^r z^{pd}\right),$$

where Exp is the plethystic exponential and τ is a rational number. Similarly $B_{r,d}$ is given by

$$\sum_{\substack{r,d \in \mathbb{Z}_{\geq 0} \\ (r,d) \neq (0,0)}} B_{r,d} w^r z^d = \mathbb{L} \operatorname{Log}\left(\Omega_X^{mot}\right).$$

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