Motivic classes of stacks in finite characteristic and applications to stacks of Higgs bundles and bundles with connections

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Introduction

Our primary goal is to compute the motivic classes of stacks of Higgs bundles and bundles with connections in finite characteristic which was earlier done in characteristic 0 by Fedorov and Soibelmans in [FSS].

Counting points of varieties over $\mathbb{F}_q$

Let $|X| := |X(\mathbb{F}_q)|$ denote the number of rational points of an algebraic variety X over a finite field $\mathbb{F}_q$ with q elements.

- $|\mathbb{A}^n| = q^n$.
- $|\mathbb{P}^n| = 1 + q + \cdots + q^n$ using cell decomposition.
- $|\mathrm{GL}(n)| = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$ considered as the space of n linearly independent columns.

Motivic classes of varieties

For any field k, one defines the abelian group $\mathrm{Mot}_{var}(k)$ as the group generated by isomorphism classes of varieties over k modulo the following relations:
1) $[X] = [Y] + [X - Y]$ where Y is a closed subvariety of X,
2) $[X] = [Y]$ if there is a radicial and surjective morphism $X \to Y$ of varieties over k.
The class $[X]$ in $\mathrm{Mot}_{var}(k)$ is called the motivic class of the variety X. We can give a ring structure for $\mathrm{Mot}_{var}(k)$ by $[X] \cdot [Y] = [X \times_k Y]$ with the unit element $[\operatorname{Spec} k] = 1$.

Realizations of the motivic classes

- Point counting $\# : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$
$$\#([X]) = |X(\mathbb{F}_q)|.$$
- Euler characteristic $\chi : \mathrm{Mot}_{var}(k) \to \mathbb{Z}$
$$\chi([X]) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q}).$$
- E-polynomial $E : \mathrm{Mot}_{var}(k) \to \mathbb{Z}[u, v]$
$$E([X]) = \sum_{k,p,q} (-1)^k \dim_{\mathbb{C}} H_c^{k,p,q}(X) u^p v^q.$$

Counting points of projective spaces

We are interested in performing computations for quotients. One of the motivation comes from the following computations for projective spaces.

- $|\mathbb{P}^n|$ can also be computed by presenting $\mathbb{P}^n$ as a quotient of $\mathbb{A}^{n+1} - \{0\}$ by the action of $\mathrm{GL}(1)$.
$$\left|\frac{\mathbb{A}^{n+1} - \{0\}}{\mathrm{GL}(1)}\right| = \frac{q^{n+1} - 1}{q - 1} = 1 + q + \cdots + q^n.$$

Counting points of groupoids

If $\mathcal{G}$ is a groupoid, we can define the volume of $\mathcal{G}$ as a weighted sum
$$|\mathcal{G}| := \sum_{G \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(G)|}.$$

- The volume of the groupoid of finite sets is e. Indeed, for any finite set of size n, the automorphism group is of order $n!$
$$|\mathcal{G}| = \sum_{G \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(G)|} = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Counting points of algebraic stacks

- We can try to do similar computations even when the group action is not free and the quotient is no longer a variety.
- Algebraic stacks generalize varieties by allowing points to have nontrivial automorphisms.
- Examples include quotients X/G of a variety by a non-free algebraic group action, moduli of curves, moduli of vector bundles, etc.

If $\mathcal{Y}$ is an algebraic stack over $\mathbb{F}_q$, we can generalize point counting for varieties by defining the volume of $\mathcal{Y}$ as a weighted sum over all its $\mathbb{F}_q$-rational points
$$|\mathcal{Y}| := \sum_{y \in \mathrm{Ob}(\mathcal{Y}(\mathbb{F}_q))} \frac{1}{|\mathrm{Aut}(y)|}.$$
This is not guaranteed to converge. However, all the stacks we consider are of finite type, for which it converges (the sum is finite).

Motivic classes of stacks

One defines the abelian group $\mathrm{Mot}(k)$ as the group generated by isomorphism classes of stacks of finite type over k modulo the following relations:
1) $[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} - \mathcal{Y}]$ where $\mathcal{Y}$ is a closed substack of $\mathcal{X}$,
2) $[\mathcal{X}] = [\mathcal{Y}]$ if there is a radicial and surjective morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over k,
3) $[\mathcal{X}] = [\mathcal{Y} \times \mathbb{A}_k^r]$ where $\mathcal{X} \to \mathcal{Y}$ is a vector bundle of rank r.
The class $[\mathcal{X}]$ in $\mathrm{Mot}(k)$ is called the motivic class of the stack $\mathcal{X}$. Similarly we can give a ring structure for $\mathrm{Mot}(k)$ by $[\mathcal{X}] \cdot [\mathcal{Y}] := [\mathcal{X} \times_k \mathcal{Y}]$.

Relation between the motivic classes

For a variety X, in [Eke] we have
$$[X/\mathrm{GL}(n)] = [X]/[\mathrm{GL}(n)].$$
Set $\mathbb{L} := [\mathbb{A}_k^1]$. There is a natural group isomorphism in [Li]
$$\mathrm{Mot}_{var}(k)[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1} | i > 0] \cong \mathrm{Mot}(k).$$

Completion of motivic class

Let $\mathrm{F}^m \mathrm{Mot}(k)$ be the subgroup generated by the classes of stacks of dimension $\le -m$. This is a ring filtration and we define the completed ring $\overline{\mathrm{Mot}}(k)$ as the completion of $\mathrm{Mot}(k)$ with respect to this filtration.

Motivic zeta function

We can define the motivic zeta function for every reduced quasi-projective variety X:
$$Z_X(t) = \sum_{n \ge 0} [Sym^n X] \cdot t^n \in 1 + t \cdot \mathrm{Mot}_{var}(k)[[t]]$$
where $Sym^n X$ is the symmetric product X^n / S_n (with the convention that $Sym^0 X = \operatorname{Spec} k$).

Fix a smooth projective geometrically connected curve X over k.

Bundles with connections

A bundle with connection on X is a pair (E, ∇) where E is a vector bundle on X and $\nabla : E \to E \otimes \Omega_X$ is a k-linear morphism of sheaves satisfying Leibniz rule, i.e. for any open subset U of X, any $f \in H^0(U, \mathcal{O}_X)$ and any $s \in H^0(U, E)$ we have
$$\nabla(fs) = f\nabla(s) + s \otimes df.$$
We use $\mathcal{C}onn_{r,pd}$ to denote the moduli stack of rank r degree pd vector bundles with connections on X.

Moduli stacks of semistable bundles

The bundle with connection (E, ∇) is called semistable if for any subbundle $F \subset E$ preserved by ∇
$$\frac{\deg F}{\operatorname{rk} F} \le \frac{\deg E}{\operatorname{rk} E}.$$

Compare the two motivic classes

Similarly we use $\mathcal{H}iggs_{r,d}^{ss}$ to denote the moduli stack of rank r degree d semistable Higgs bundles on X. For the characteristic 0 case in [FSS], we have
$$[\mathcal{C}onn_r(X)] = [\mathcal{H}iggs_{r,0}^{ss}(X)].$$

Formula for bundles with connections

For finite characteristic p and sufficiently large e, we have $[\mathcal{C}onn_{r,pd}^{ss}] = C_{r,p(d+er)}$, where $C_{r,pd}$ is defined via the generating function
$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} C_{r,pd} w^r z^{pd} = \mathrm{Exp}\left(\sum_{d/r=\tau} B_{r,pd} w^r z^{pd}\right),$$
where τ is any rational number, Exp is the plethystic exponential and $B_{r,d}$ is a Donaldson-Thomas invariants in $\overline{\mathrm{Mot}}(k)$.

References

[Eke] T. Ekedahl. The Grothendieck group of algebraic stacks. *ArXiv e-prints*, March 2009.
[FSS] R. Fedorov, A. Soibelman, and Y. Soibelman. Motivic classes of moduli of Higgs bundles and moduli of bundles with connections. *Communications in Number Theory and Physics.* 12(4):687-766, 2018.
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