MODULI SPACE OF GENERALIZED LINE BUNDLES OF REDUCIBLE CURVES

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ABSTRACT. This note is a summary for several papers [1][2][3] about moduli space of generalized line bundles, and we focus on the degree 0 case of reducible curves regarded as Kodaira fiber of type I_2.

CONTENTS

1. INTRODUCTION

Let C be a projective curve over k with an ample invertible sheaf L, and H is the associated polarization. Let h denote the degree of H.

Definition 1.1. (1) A sheaf F on C is pure dimension one if the support of any nonzero subsheaf of F is of dimension one. The rank and degree with respect to H of F are the rational number $r_H(F)$ and $d_H(F)$ determined by the Hilbert polynomial

$$P(F, n, H) = \chi(F \otimes \mathcal{O}_C(nH)) = hr_H(F)n + d_H(F) + r_H(F)\chi(\mathcal{O}_C).$$

(2) The slope of F is defined by

$$\mu_H(F) = \frac{d_H(F)}{r_H(F)}$$

(3) The sheaf is (semi)stable with respect to H if F is pure of dimension one and for any proper subsheaf $F' \hookrightarrow F$ one has

$$\mu_H(F') < (\leq)\mu_H(F)$$

(4) In Simpson's paper we define the multiplicity of F as the integer number $hr_H(F)$ and the Simpson's slope as the quotient

$$\frac{d_H(F) + r_H(F)\chi(\mathcal{O}_C)}{hr_H(F)}$$

Remark 1.2. Stability and semistability considered in terms of Simpson's slope and interms of μ_H are equivalent.

Definition 1.3. For every semistable sheaf F with respect to H there is a Jordan-Holder filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = F$$

with stable quotients F_i/F_{i-1} and $\mu_H(F_i/F_{i-1}) = \mu_H(F)$ for $i = 1, \ldots, n$. This filtration may not be unique, but the graded object $Gr(F) = \sum_i F_i/F_{i-1}$ is independent on the choice of the filtration. Two semistable sheaves F, F' are said to be S-equivalent if $Gr(F) \simeq Gr(F')$, denoting their class by $[F]$.

Definition 1.4. (1) We denote $\mathrm{Jac}^0_{s(s)}(C)$ of isomorphism classes of (semi)stable invertible sheaves with degree 0 with respect to the polarization.
(2) A sheaf F is called generalized line bundle on C if it is of pure dimension one with rank 1 on every irreducible component of C.
(3) We also denote $\overline{\mathrm{Jac}}^0_{ss}(C)$ the space of isomorphism classes of degree 0 semistable generalized line bundle.

Definition 1.5. If F is a pure dimension one sheaf on C, for every proper subcurve D of C, we define $F_D = (F \otimes \mathcal{O}_D)/torsion$, π_D is the surjective morphism $F \to F_D$ and $F^D = \ker \pi_D$. Denote h_D the degree of H_D on D, and simply write $d = d_H(F)$, $d_D = d_{H_D}(F_D)$. The closure of $C - D$ will be denoted by $\overline{D}$. If $g = g(C)$ denotes the arithmetic genus of C, for any pure dimension one sheaf F on X of polarized rank 1 and degree d with respect to H, let b, $0 \le b < h$ be the residue class of $d - g$ modulo h so that

$$d - g = ht + b.$$

For every proper subcurve D of C, we shall write

$$k_D = \frac{h_D(b+1)}{h}.$$

2. Main Theorems and Results

If C is a curve of type I_2 , then $C = C_1 \cup C_2$ with $C_1 \cdot C_2 = P + Q$. The irreducible components C_1, C_2 are rational smooth curves. For the degree 0 line bundles on C, there is a lemma

Lemma 2.1. *Let L be a degree 0 line bundle, then L is (semi)stable with respect to polarization H if and only if for C_1, C_2 the following inequalities hold:*

$$-1 < (\le) d_{C_i} < (\le) 1.$$

With this lemma, we have the following

Proposition 2.2. *(1) Let $C = C_1 \cup C_2$ be a curve of type I_2, there is an exact sequence*

$$0 \to k^* \to \mathrm{Pic}(C) \to \prod_{i=1}^{2} \mathrm{Pic}(C_i) \to 0.$$

(2) Furthermore, let H be a polarization on C, then for the Simpson Jacobian of C of degree 0 we have the following exact sequence (the other case is the same but the role of C_1, C_2 be interwined.)

$$0 \to k^* \to \mathrm{Jac}^0_s(C) \to \prod_{i=1}^{2} \mathrm{Pic}^0(C_i) \to 0, \ \textit{and}$$

$$\mathrm{Jac}^0_{ss}(C) - \mathrm{Jac}^0_s(C) = \mathrm{Pic}^{-1}(C_1) \prod \mathrm{Pic}^1(C_2).$$

With proposition above, we can collect more properties of degree 0 semistable generalized line bundles on C

Theorem 2.3. *If $C = C_1 \cup C_2$ is a curve of type I_2, it holds that:*
(1) The (semi)stability of a degree 0 generalized line bundle on C does not depend on the polarization.
(2) A degree 0 line bundle L on C is stable if and only if $L|_{C_i} \simeq \mathcal{O}_{C_i}$ for all i.
(3) A degree 0 line bundle L on C is semistable if and only if $L|_{C_1} \simeq \mathcal{O}_{C_1}(1)$, $L|_{C_2} \simeq \mathcal{O}_{C_1}(-1)$ (the other case is the same but with the roles of C_1 and C_2 interwined).
(4) If F is degree 0 stable generalized line bundle. on C, then it is a line bundle.
(5) If F is a degree 0 semistable generalized line bundle. on C, then its graded object is $Gr(F) = \oplus_{i=1}^{2} \mathcal{O}_{C_i}(-1)$.

In order to construct the moduli space $\overline{\mathrm{Jac}}_{ss}^{0}(C)$, let q be a fixed smooth point of C and denote C_1 the irreducible component of C containing q. If $\Delta \subset C \times C$ denotes the diagonal and I_Δ is its ideal sheaf, define $\mathcal{O}_{C\times C}(\Delta) = Hom(I_\Delta, \mathcal{O}_{C\times C})$ as the dual of I_Δ. Consider the sheaf

$$\mathcal{E} = \mathcal{O}_{C\times C}(\Delta) \otimes \pi_1^* \mathcal{O}_C(-q)$$

where $\pi_1 : C \times C \to C$ is the projection on the first component. This sheaf is flat over C via the projection $\pi_2 : C \times C \to C$ and we have the following

Theorem 2.4. *(1) For any point $p \in C$, the restriction $\mathcal{E}_p$ of $\mathcal{E}$ to $C \times \{p\}$ is a semistable pure dimension one sheaf of rank 1 and degree 0. Moreover, if p is not a smooth point of C_1, then all sheaves $\mathcal{E}_p$ define the same point of $\overline{\mathrm{Jac}}_{ss}^{0}(C)$.*
(2) The restriction of the family $\mathcal{E}$ to $C \times C_1$ gives, by the universal property of $\overline{\mathrm{Jac}}_{ss}^{0}(C)$, a map

$$\phi : C_1 \to \overline{\mathrm{Jac}}_{ss}^{0}(C)$$

defined as $\phi(p) = [\mathcal{E}_p]$. Indeed, this is a surjection and $\overline{\mathrm{Jac}}_{ss}^{0}(C)$ is a rational curve with one node.

3. Proof of Lemma 2.1.

Lemma 3.1. *Let F be a pure dimension one sheaf on C supported on a subcurve D of X. Then F is (semi)stable with respect to H_D if and only if F is (semi)stable with respect to H.*

Proof. It follows from the equality

$$P(F, n, H) = \chi(i_* F \otimes \mathcal{O}_C(nH)) = \chi(F \otimes \mathcal{O}_D(nH_D)) = P(F, n, H_D)$$

where $i : D \hookrightarrow C$ is the inclusion map. □

Lemma 3.2. *A torsion free rank 1 sheaf F on C is (semi)stable if and only if $\mu_H(F^D) < (\leq)\mu_H(F)$ for every proper subcurve D of C.*

Proof. Given a subsheaf G of F such that $Supp(G) = D \subset C$, let us consider the complementary subcurve $\overline{D}$ of D in C. Since $F_{\overline{D}}$ is torsion free, we have $G \subset F^{\overline{D}}$ with $r_H(G) = r_H(F^{\overline{D}})$ so that $\mu_H(G) \leq \mu_H(F^{\overline{D}})$ and the result follows. □

Proof of Lemma 2.1. Write $d = g + ht + b$. If L is (semi)stable with respect to H and D is a proper sucurve of X, the condition $\mu_H(L^D) < (\leq)\mu_H(L)$ is equivalent to

$$-\chi(\mathcal{O}_D) + h_D t + k_D < (\leq) d_D.$$

Similarly for subsheaf $L^{\overline{D}}$

$$-\chi(\mathcal{O}_{\overline{D}}) + h_{\overline{D}}t + k_{\overline{D}} < (\leq) d_D.$$

Since $C = D \cup \overline{D}$ and $\alpha_D = D \cdot \overline{D}$, we have $d = d_D + d_{\overline{D}}$, $h = h_D + h_{\overline{D}}$ and $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_D) + \chi(\mathcal{O}_{\overline{D}}) - \alpha_D$. In this special case $d = 0$, $g(X) = 1$, we have $b = h - 1$ and $t = -1$. Now $k_{C_i} = h_{C_i}$. Then the above inequalities give the desired ones. Conversely, this is true for any connected subcurve D of C, by Lemma 3.2 this is true. □

4. Proof of Proposition 2.2.

Corollary 4.1. *Let $C = C_1 \cup C_2$ be a projective reduced and connected curve over k. Suppose that the intersection points P, Q of its irreducible components are ordinary double points. Let $C' = C_1 \coprod C_2$ be the partial normalization of C at the nodes P, Q. Then there is an exact sequence*

$$0 \to k^* \to \mathrm{Pic}(C) \to \mathrm{Pic}(C') \to 0.$$

Proof. This is an consequence of a proposition due to Grothendieck ([4], Prop, 21.8.5). □

Proof of Proposition 2.2.
(1) With Theorem 4.5. in [1] and Corollary 4.1., we can prove this part.
(2) By Lemma 2.1, L is stable if and only if

$$-1 < d_{C_i} < 1.$$

For the integer d_{C_i}, it must be 0. By the first part we have the exact sequence. Suppose now that the sheaf L is strictly semistable, d_{C_1} is -1 or 1. Assume that $d_{C_1} = -1$. Then $\mu_H(L_{C_1}) = \mu_H(L^{C_1}) = 0$ and L_{C_1}, L^{C_1} are stable sheaves with respect to H_{C_1} and H_{C_2}. Thus, $L^{C_1} \subset L$ is a Jordan-Holder filtration for L, the S-equivalence class belongs to $\mathrm{Pic}^{-1}(C_1) \prod \mathrm{Pic}^1(C_2)$. Hence, there is only one S-equivalence class of strictly semistable line bundles. □

5. Proof of Theorem 2.3.

Proof of Theorem 2.3.
(2)(3) The result is straightforward by Proposition 2.2.
(4) If F is a stable generalized line bundle but not a line bundle, by Proposition 5.15 in [1], $F = \phi_*(G)$ where G is a stable pure dimension one sheaf of rank 1 and degree -1 on curve C'. By Lemma 6.1. in [1], it is impossible.
(1) For line bundles, the result follows from (2)(3)(4). If F is not a line bundle, by Lemma 3.2, it is semistable if and only if $-\chi(\mathcal{O}_D) \leq d_D$ for any $D \subset C$, which does not depend on the polarization since

$$d_D = d_{H_D}(F_D) = \chi(F_D) - \chi(\mathcal{O}_D)$$

(5) If F is a degree 0 semistable generalized line bundle on C, by Proposition 5.15. and Example 4.7. in [1], we have

$$\overline{\mathrm{Jac}}^0_{ss}(C) - \mathrm{Jac}^0_{ss}(C) \cong \overline{\mathrm{Jac}}^0_{ss} C' \cong \prod_{i=1}^{2} \mathrm{Pic}^{-1}(C_i). \quad \square$$

6. Proof of Theorem 2.4.

Proof of Theorem 2.4.
(1) Since C is Gorestein, we have that

$$\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{O}_C(-q)) = k,$$

so that the restriction $\mathcal{E}_p$ is the unique non trivial extension

$$0 \to \mathcal{O}_C(-q) \to \mathcal{E}_p \to \mathcal{O}_p \to 0.$$

Using this exact sequence, $\mathcal{E}_p = I_p^* \otimes \mathcal{O}_C(-q)$ is a degree 0 generalized line bundle. If p is a smooth point of C, we have $\mathcal{E}_p = \mathcal{O}(p-q)$.
(i)If $p \in C_1$, $\mathcal{E}_p$ restrict to C_1, C_2 are of degree 0, by Theorem 2.3, it is stable.
(ii) If $p \notin C_1$, it is in C_2, since the restriction of $\mathcal{E}_p$ to C_1 has degree -1, to C_2 degree 1, by Theorem 2.3, it is strictly semistable.
(iii) If p is a singular point of C, $\mathcal{E}_p$ is not invertible and by Theorem 2.3, it is not stable. Let $\mathcal{G} \hookrightarrow \mathcal{E}_p$ be a proper subsheaf, by Lemma 2.1, the line bundle $\mathcal{O}_C(-q)$ is stable. Consider the composition map

$$g : \mathcal{G} \to \mathcal{E}_p \to \mathcal{O}_p,$$

it is either zero or surjective. If g is zero,we have $\mathcal{G} \hookrightarrow \mathcal{O}_C(-q)$ and then $\mu_H(\mathcal{G}) < -1$. If g is surjective, denote its kernel by $\mathcal{H}$, it is a subsheaf of $\mathcal{O}_C(-q)$ and then $\mu_H(\mathcal{H}) < -1$. Since the degree of $\mathcal{H}$ is integer, from

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{O}_p \to 0,$$

we conclude that $\mu_H(\mathcal{G}) \leq 0$. Finally $\mathcal{E}_p$ is a strictly semistable sheaf.
(iv) For the second part, if p is not smooth point of C_1, then $\mathcal{E}_p$ is a strictly semistable sheaf. By Theorem 2.3, its graded object is isomorphic to $\prod_{i=1}^{2} \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence they are in the same S-equivalence class.
(2) In this case the intersection points P, Q are sent to the same class in $\overline{\mathrm{Jac}}_{ss}^0(C)$. By Theorem 2.3, there is only one extra point in $\overline{\mathrm{Jac}}_{ss}^0(C)$ corresponds to any strictly semistable sheaf, we can conclude that it is surjective and $\overline{\mathrm{Jac}}_{ss}^0(C)$ is a rational curve with one node.

Remark 6.1. There are two different proofs for (1), Lemma 2.17 in [3] and Lemma 6.3.4 in [5].

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