Motivation for the Introduction of Stacks

Moduli Spaces: Motivation

- **What is a moduli problem?**
- Philosophically speaking a moduli problem is a classification problem. In geometry or topology, for example, we like to classify interesting geometric objects like manifolds, algebraic varieties, vector bundles or principal G-bundles up to their intrinsic symmetries, i.e. up to their isomorphisms depending on the particular geometric nature of the objects.

- Just looking at the set of isomorphism classes of the geometric objects we like to classify normally does not give much of an insight into the geometry. To solve a moduli problem means to construct a certain **geometric object**, a moduli space, which could be for example a topological space, a manifold or an algebraic variety such that **its set of points corresponds bijectively to the set of isomorphism classes** of the geometric objects we like to classify.

- We could therefore say that a moduli space is a solution space of a given classification problem or moduli problem. In constructing such a moduli space we obtain basically a parametrizing space in which the **geometric objects** we like to classify **are** then parametrized by the **coordinates of the moduli space**.

Moduli Spaces : Motivation

- **However:**
- Constructing a moduli space as the solution space for a given moduli problem is normally not all what we like to ask for.
- We also would like to have a way of understanding how the different isomorphism classes of the geometric objects can be constructed geometrically in a **universal** manner.
- So what we really like to construct is a **universal geometric object**, such that all the other geometric objects can be constructed from this universal object in a kind of unifying way

Classification of Vectorbundles as an Example

- We like to study the moduli problem of classifying vector bundles of fixed rank over an algebraic curve over a field k.
- Let X be a smooth projective algebraic curve of genus g over a field k.
- We define the moduli functor M_X^n as the contravariant functor from the category (Sch/k) of all schemes over k to the category of sets

$$M_X^n\colon (\text{Sch}/k)^{\text{op}} \to (\text{Sets}).$$

Example: Vectorbundles

- On objects the functor M_X^n is defined by associating to a scheme U in (Sch/k) the set M_X^n(U) of isomorphism classes of families of vector bundles of rank n on X parameterized by U, i.e. the set of isomorphism classes of vector bundles E of rank n on X × U.
- On morphisms M_X^n is defined by associating to a morphism of schemes f : $U_0 \to$ U the map of sets f^* : M_X^n (U) → M_X^n (U_0) induced by pullback of the vector bundle E along the morphism $id_X \times f$ as given by the commutative diagram.
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$$\begin{tikzcd}
(id_X \times f)^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
X \times U' \arrow[r, "id_X \times f"] & X \times U
\end{tikzcd}$$

Example: Vectorbundles

- The moduli problem for classifying vector bundles of rank n and degree d on a smooth projective algebraic curve X is now equivalent to the following question.
- Is the moduli functor M_X^n representable? In other words, does there exist a scheme M_n in the category (Sch/k) such that for all schemes U in (Sch/k) there is a bijective correspondence of sets
- M_n (U) $\cong$Hom(Sch/k)(U, M_n)?

- If such a scheme M_n exists, it is also called a fine moduli space

Example: Vectorbundles

- Now let us assume that this functor is representable by a scheme M_n . We then have
- M_X^n (U) ≘ Hom(Sch/k)(U, M_n)
- If a fine moduli space M_n exists, we would have in particular a bijective correspondence
- M_X^n (Spec(k)) ≘ Hom(Sch/k)(Spec(k), M_n) = |M_n|

- But this means that isomorphism classes of vector bundles over X are in bijective correspondence with points of the moduli space M_n .

Example: Vectorbundles

- If a fine moduli space M_n exists, we would also have a bijective correspondence
- M_X^n $(M_n) \cong \text{Hom(Sch/k)}(M_n, M_n)$
- Now let E^{univ} be the element of the set M_X^n (M_n) corresponding to the morphism id_{Mn} , i.e. E^{univ} is a vector bundle of rank n over $X \times M_n$.

Example: Vectorbundles

- This vector bundle E^{univ} over $X \times M_n$ is called a **universal family** of vector bundles over X, because representability implies that for any vector bundle E over $X \times U$ there is a **unique** morphism

$f : U \rightarrow M_n$ such that $E \cong (id_X \times f)^*(E^{univ})$ in the pullback diagram

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$$\begin{tikzcd}
\mathcal{E} \cong (id_X \times f)^* \mathcal{E}^{univ} \arrow[r] \arrow[d] & \mathcal{E}^{univ} \arrow[d] \\
X \times U \arrow[r, "id_X \times f"] & X \times M_X^{n,d}
\end{tikzcd}$$

Example: Vectorbundles

- Representability of the moduli functor M_n would therefore solve the moduli problem and addresses both desired properties of the solution, namely the existence of a geometric object such that its points correspond bijectively to isomorphism classes of vector bundles on the curve X and the existence of a universal family E^{univ} of vector bundles such that any family of vector bundles E over X can be constructed up to isomorphism as the pullback of the universal family E^{univ} along the classifying morphism

Problems

It unfortunately turns out that most moduli problems do not admit fine moduli spaces, i.e their corresponding moduli functors turn out not to be representable.

This, in particular, also holds for our example at hand, the classification of vector bundles over smooth curves as we will shortly see.

The Moduli Functor M_X^n is not representable

- We can argue as follows to show that the moduli functor M_X^n is not representable:
- Let E be a vector bundle on X × U and let $pr_2 : X \times U \to U$ be the projection map. In addition, let L be a line bundle on U.
- Define the induced bundle $E_0 := E \otimes pr^*{}_2 L$.
- As vector bundles are always locally trivial in the Zariski topology it follows that there exists an open covering {Ui}i∈I of the scheme U such that the restriction $L|_{Ui}$ of L on Ui is the trivial bundle for all i ∈ I.

The Moduli Functor M_X^n is not representable

- We will have on X × Ui therefore that $E|_{X\times Ui} \cong E_0|_{X\times Ui}$. Assume now that the moduli functor M_X^n is representable , i.e. there exists a scheme M_n such that for all schemes U in the category (Sch/k) there is a bijective correspondence of sets
- $M_X^n(U) \cong \mathrm{Hom}(Sch/k)(U, M_n)$
- Then it follows that there exists morphisms of schemes $\alpha, \alpha_0 : U \to M_n$ corresponding to the two vector bundles E and E_0 on X × U. But from the remarks above on local triviality of vector bundles it follows that the restrictions of α and α_0 on Ui must be equal for all $i \in I$,
- i.e. $\alpha|_{Ui} = \alpha_0|_{Ui}$
- And from this it would follow immediately that $\alpha = \alpha_0$ and therefore $E \cong E_0$.
- But in general the two vector bundles E and E_0 are not necessarily globally isomorphic

Alternatives

- There are basically two approaches to circumvent the problem of non-representability of the moduli functor:

1. Restrict the class of vector bundles to be classified to eliminate automorphisms, i.e. rigidify the moduli problem via restriction of the moduli functor to a smaller class of vector bundles and use a weaker notion of representability.
2. Record the information about automorphisms by organizing the moduli data differently, i.e. enlarge the category of schemes to ensure representability of the moduli functor

Towards Stacks

- Let us briefly discuss how this second approach applies to our motivating example, the moduli problem of vector bundles of rank n on a smooth projective algebraic curve X.
- How can we record the moduli data differently so that we don’t lose the information from the automorphisms?
- Instead of passing to sets of isomorphism classes of vector bundles we will use a categorical approach to record the information coming from the automorphisms.

Towards Stacks

- As above let X be a smooth projective algebraic curve of genus g over a field k.
- We define the moduli stack Bun^n as the contravariant “functor” from the category (Sch/k) of schemes over k to the category of groupoids Grpds
- $Bun^n : (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Grpds}$

Towards Stacks

- On objects Bun^n is defined by associating to a scheme U in (Sch/k) the category $Bun^n(U)$ with objects being vector bundles E of rank n on $X \times U$ and morphisms being vector bundle isomorphism, i.e. for every scheme U the category $Bun^n(U)$ is a groupoid, i.e. a category in which all its morphisms are isomorphisms.
- On morphisms Bun^n is defined by associating to a morphism of schemes f : $U_0 \to U$ a functor $f^* : Bun^n(U) \to Bun^n(U_0)$ induced by pullback of the vector bundle E along the morphism $id_X \times f$ as given by the pullback diagram

$$\begin{tikzcd}
(id_X \times f)^*\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
X \times U' \arrow[r, "id_X \times f"] & X \times U
\end{tikzcd}$$

Towards Stacks

- Because pullbacks are only given up to natural isomorphisms we also have for any pair of composable morphisms of schemes $U_1 \to U_0 \to U$ a natural isomorphism between the induced pullback functors
- $g^* \circ f^* \cong (f \circ g)^*$
- And these natural isomorphisms will be associative with respect to composition.
- Notice: Bun^n is not really a "functor" in the classical categorical sense as it preserves composition only up to specified isomorphisms and Bun^n is therefore what in general is called a pseudo-functor.

Towards Stacks

- An important feature of vector bundles is that they have the special property that they can be defined on open coverings and glued together when they are isomorphic when restricted to intersections.
- what we really will get here for Bun^n is a pseudo-functor with glueing properties on the category (Sch/k) once we have specified a topology called Grothendieck topology on the category (Sch/k) in order to be able to speak of "coverings" and "glueing ".
- Such pseudo-functors with glueing properties, like Bun^n are called stacks

Summary:Why Stacks

- The moduli problem of classifying vector bundles of rank n over a smooth projective algebraic curve X of genus g over the field k has no solution in the category (Sch/k), but in stacks.
- *This means, Bun^n* will be representable in stacks

- Another motivation for the introduction of stacks are quotient problems, i.e. quotients of schemes by algebraic groups (alternative to GIT approach)

Grothendieck topology

C - category with fiber products

Grothendieck topology on C is given by a function τ which assigns to each object U of C a covering collection $\tau(U)$ consisting of families of morphisms $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ with target U such that:

1. (Isomorphisms) If $U' \to \boxed{U}$ is an isomorphism, then $\{U' \to U\} \in \tau(U)$

2. (Transitivity) If $\{U_i \xrightarrow{\varphi_i} \underline{U}\}_{i \in I} \in \tau(U)$, and $\forall\, i \in I$, $\{U_{ij} \xrightarrow{\varphi_{ij}} U_i\}_{j \in J} \in \tau(U_i)$,

then $\{U_{ij} \xrightarrow{\varphi_i \circ \varphi_{ij}} U\}_{i,j \in J} \in \tau(U)$

3. (Base change) If $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I} \in \tau(U)$ and $V \hookrightarrow U$ is any morphism, then

$$\{V \underset{U}{\times} U_i \longrightarrow V\}_{i \in I} \in \tau(V)$$

$\to V \cap U_i$

- Families in $\tau(U)$ are called covering families for U in the τ - topology.
- A category with a topology τ is called a site and is denoted by C_τ.

Before coming to important examples of Grothendieck topologies on Sch/S, we recall some morphisms of schemes.

$f: X \to Y$ $x \in X$

- Locally of finite type $\overset{\cdot}{x}$ $U = \operatorname{Spec} A$
- Locally of finite presentation

$B \longrightarrow A$ $\operatorname{Spec} B$ $\cdot f(x)$

A into a f.g. B-alg.

- Finite type.
- A morphism $f: X \longrightarrow Y$ of schemes is an open (resp. closed) embedding if it factors into an isomorphism $X \longrightarrow Y'$ followed by an inclusion $Y' \hookrightarrow Y$ of an open (resp. closed) subscheme Y' of Y.

- Quasi-compact: Pre-image of compact open is compact open.
- Separated: $\Delta : X \longrightarrow X \times_Y Y$ is a closed embedding.
- Quasi-separated: Δ is quasi-compact
- Proper: Separated, finite type and $\underbrace{\text{universally closed}}$.

$\forall\ Z \longrightarrow Y$, $Z \times_Y X \longrightarrow X$

$$\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

closed map

$\mathbb{A}^1_k$ closed but not universally Spec k closed.

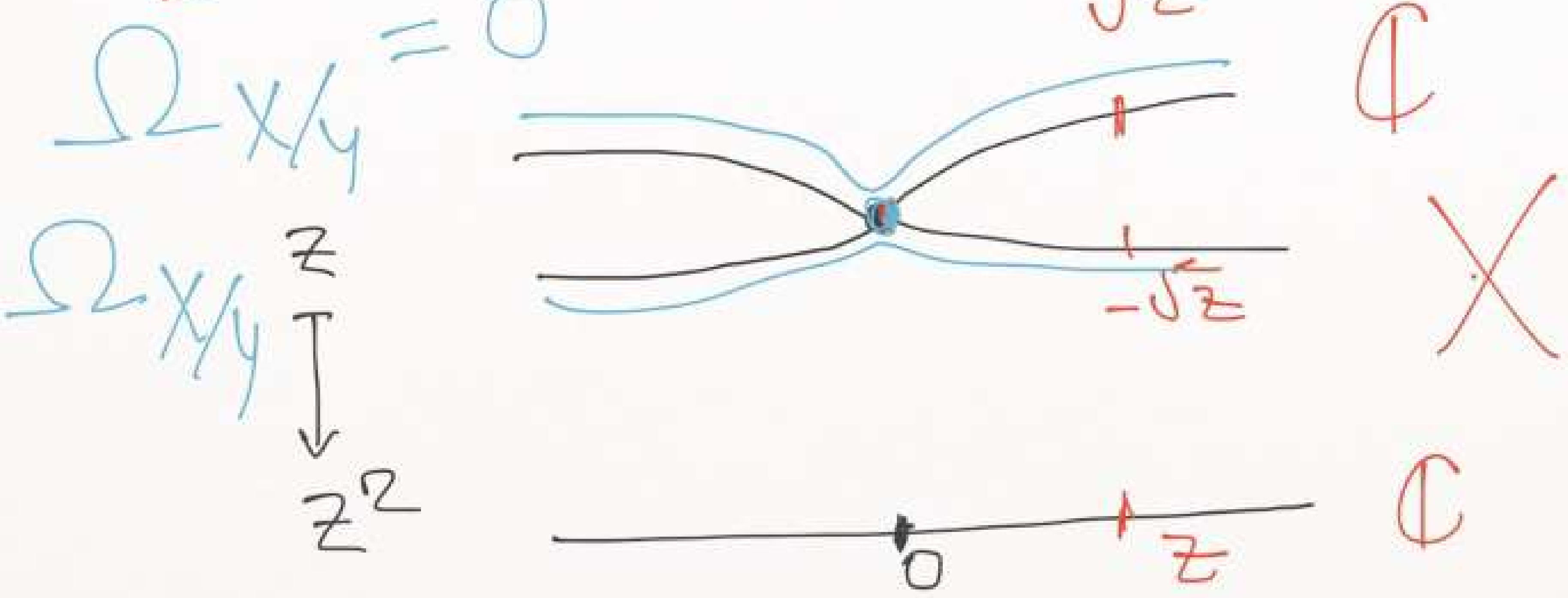
- Flat: $\forall x \in X$, the map $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{Y,f(x)}$-module.

- Faithfully flat: Flat + surjective
- fppf: Faithfully flat + Locally of finite presentation
- fpqc: Faithfully flat + every quasi-compact open of Y is the image of a quasi-compact open of X.

not exactly q.c.

This notion of fpqc includes Zariski cover

- Etale: Flat + unramified



Unramified : lfp + ⊛

$\forall x \in X$, consider the map

$$f^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

$\mathfrak{m} :=$ maximal ideal of $\mathcal{O}_{Y,f(x)}$

$\mathfrak{n} := f^{\#}(\mathfrak{m}) =$ ideal generated by $\mathfrak{m}$ in $\mathcal{O}_{X,x}$.

⊛ : $\mathfrak{n}$ is actually the maximal ideal in $\mathcal{O}_{X,x}$ and the map

$$\mathcal{O}_{Y,f(x)}/\mathfrak{m} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{n}$$

is a finite, separable field extension.

- Smooth: Lfp + flat + for any morphism $\mathrm{Spec}(k) \longrightarrow Y$ w/ $\bar{k} = k$, the geometric fiber $X \times_Y \mathrm{Spec}(k)$ is regular scheme/ i.e., all local rings are regular

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Examples of sites: → small site
→ big site

X scheme over a base scheme S.

- $\overline{X}_{Zar}$: objects: $U \longrightarrow X$ open embedding

  morphisms: morphisms $U' \longrightarrow U$ over X

$$\begin{tikzcd} U' \arrow[rr] \arrow[dr] & \circlearrowright & U \arrow[dl] \\ & X & \end{tikzcd}$$

$\tau(U)$ consists of $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ if $\bigcup_{i\in I} \varphi_i(U_i) = U$.

$X_{\underline{Zar}}$ = small Zariski site.

- $X_{\underline{\acute{e}t}}$: Objects : $U \xrightarrow[\text{etale}]{} X$

  morphisms: etale morphisms $U' \longrightarrow U$ over X

$\tau(U)$ consists of $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ if $\bigcup_{i\in I} \varphi_i(U_i) = U$.

$X_{\acute{e}t}$ = small etale site of X.

Sch/X Sch/S

- Replacing etale by smooth (resp. fppf, resp. fpqc), we get X_{sm} (resp. X_{fppf}, resp. X_{fpqc})

$$\boxed{X_{Zar} \lneq X_{et} \lneq X_{sm} \lneq X_{fppf} \lneq X_{fpqc}}$$

- Big sites: Our category here is Sch/S.

For $U \in Sch/S$, $\tau(U)$ consists of $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ with $\bigcup_{i\in I} \varphi_i(U_i) = U$ of open embeddings (resp. etale, resp. smooth, resp. fppf, resp. fpqc morphisms) to get big Zariski site $(Sch/S)_{Zar}$ (resp. big etale site $(Sch/S)_{et}$, ...).

Defn: Let C be a category. A presheaf of sets on C is a contravariant functor $\mathcal{F} : C^{op} \longrightarrow (\text{Sets})$. Morphism of presheaves of sets are given by natural transformation of functors.

Sheaf on a site: Let C_τ be a site. A presheaf of sets $\mathcal{F}$ is a sheaf if:

1. $\forall\, U \in Ob(C)$, $\forall\, f, g \in \mathcal{F}(U)$ and $\forall\, \{U_i \xrightarrow{\varphi_i} U\}_{i \in I} \in \tau(U)$, we have

identity $\boxed{f|_{U_i} = g|_{U_i}}\ \forall i \Rightarrow f = g$.

$\mathcal{F}(U) \xrightarrow{\varphi_i^*} \mathcal{F}(U_i)$ $\quad$ $f|_{U_i} := \varphi_i^*(f)$

2. $\forall \{U_i \xrightarrow{\varphi_i} U\}_{i\in I} \in \tau(U)$ and $\forall \{f_i \in \mathcal{F}(U_i)\}_{i\in I}$ such that $f_i|_{U_i \times_U U_j} = f_j|_{U_i \times_U U_j}$, $\exists\, f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$.

Defn: An S-space wrt the Grothendieck topology τ is a sheaf of sets over the site $(Sch/S)_\tau$.

Notation: Category of S-spaces

$$\| $$

$$(Spaces/S)$$

For $X \in Sch/S$, define the functor

$$\underbrace{h_X}_{\text{"functor of points"}} : (Sch/S)^{op} \longrightarrow (\text{sets})$$

$$Y \longmapsto Hom_S(Y, X)$$

Theorem (Grothendieck): For any S-scheme X, the functor

$$h_X : (Sch/S)^{op} \longrightarrow (\text{Sets})$$

is a sheaf for the fpqc topology (and hence for Zariski, etale, smooth and fppf topology).

Yoneda: $(\mathrm{Sch}/S) \hookrightarrow (\mathrm{Spaces}/S)$

- An S-space $\mathcal{F}$ is representable if $\exists$ an S-scheme X such that $\mathcal{F} \simeq h_X$.

- Algebraic spaces (Special S-space in étale topology)

Defn: An equivalence relation in the category (Spaces/S) on the big-etale site is given by two S-spaces R and X together w/ a monomorphism of S-spaces $\Leftrightarrow$ injective at the level of points

$$\delta : R \longrightarrow X \times_S X$$

mono. $\Updownarrow$ $\delta(U)$ is injective $\forall U$.

such that $\forall\, U \in (Sch/_S)$ $\delta(U)(R(U)) \subset X(U) \times X(U)$ is an equivalence relation.

Defn: A quotient S-space for such an equivalence relation is given by the coequalizer of

$$\begin{tikzcd} R \arrow[r, shift left, "pr_2 \circ \delta"] \arrow[r, shift right, "pr_1 \circ \delta"'] & X \end{tikzcd}.$$

Ex: X set, $R \subset X \times X$ equivalence relation

$$\begin{tikzcd} R \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X \arrow[r] & X/_R \end{tikzcd}$$

Defn: An algebraic space is an S-space $\mathcal{X}$ on $(Sch/S)_{ét}$ such that:

1. $\forall\ X, Y \in (Sch/S)$ and morphisms $x : X \to \mathcal{X}$, $y : Y \to \mathcal{X}$, the sheaf $X \times_{\mathcal{X}} Y$ is representable by an S-schemes.

2. $\exists$ a scheme X → atlas of $\mathcal{X}$ and a surjective étale morphism $x : X \longrightarrow \mathcal{X}$, i.e., $\forall$ morphism $y : Y \longrightarrow \mathcal{X}$ w/ Y a scheme, the projective $X \times_{\mathcal{X}} Y \longrightarrow Y$ is a surjective étale morphism of schemes

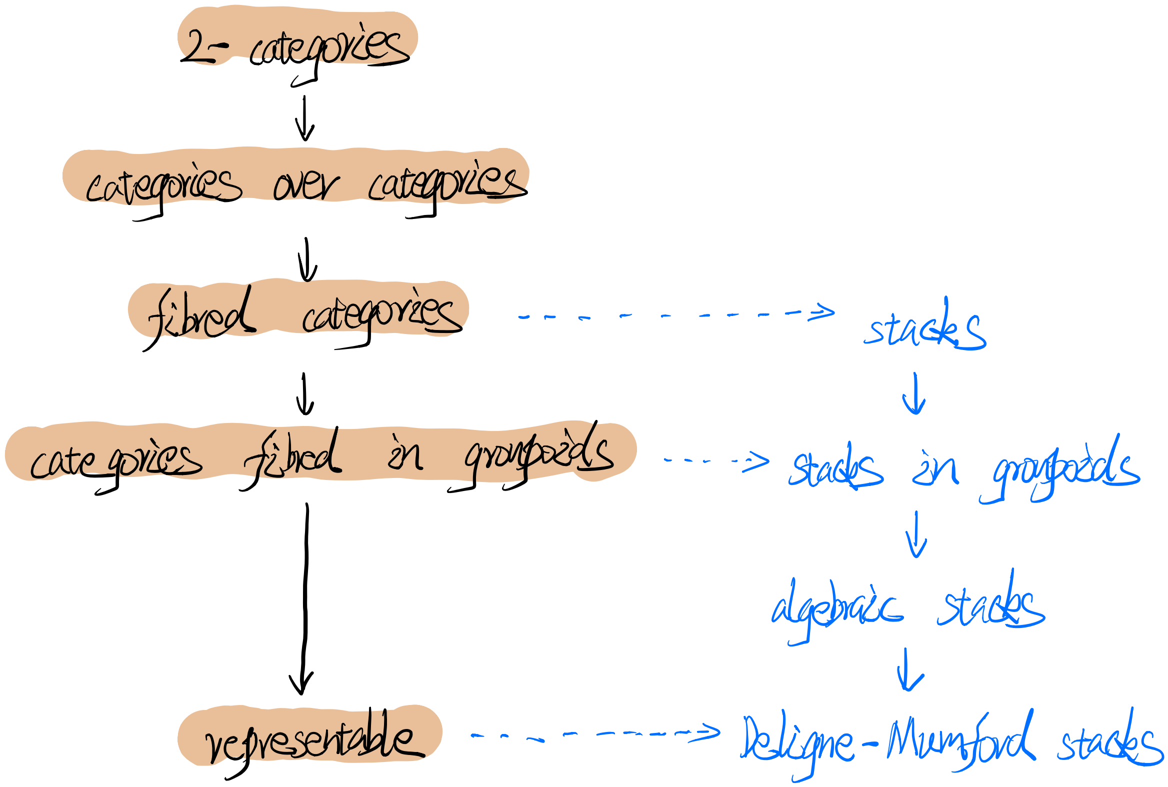
$$\begin{tikzcd}
X \times_{\mathcal{X}} Y \arrow[r, "\text{surjective étale}"] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & \mathcal{X}
\end{tikzcd}$$

Prpn: An S-space is an algebraic space iff it is the quotient S-space for an equivalence relation w/ both R and X as S-schemes, $pr_2 \circ \delta$, $pr_1 \circ \delta$ étale morphisms and δ a quasi compact morphism in (Sch/S).

→ in the étale topology.

Alg. spaces look like affine schemes in the étale topology.

$f : X \longrightarrow Y$ is of finite type if $\forall \operatorname{Spec} B \subseteq Y$, $f^{-1}(\operatorname{Spec} B)$ can be covered by finitely many Spec(f.g. B-algebra)



Reference. The Stacks project 4.28 - 4.40

I. (strict) 2-category

First we want to find the composition of 2-morphisms (now we only call them the composition of transformation of functors)

Let us denote $\mathrm{Ob}(Cat)$ the class of all categories. For every pair of categories $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(Cat)$ we have the "small" category of functors $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$. Composition of transformation of functors such as

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=60, "F''"{name=U}] \arrow[r, "F'"'{name=M}] \arrow[r, bend right=60, "F"'{name=D}] & \mathcal{B}
\arrow[Rightarrow, from=U, to=M, "t'"] \arrow[Rightarrow, from=M, to=D, "t"]
\end{tikzcd}
\text{ composes to }
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=40, "F''"{name=U}] \arrow[r, bend right=40, "F"'{name=D}] & \mathcal{B}
\arrow[Rightarrow, from=U, to=D, "t \circ t'"]
\end{tikzcd}$$

is called *vertical* composition. We will use the usual symbol $\circ$ for this. Next, we will define *horizontal* composition. In order to do this we explain a bit more of the structure at hand.

G F $G\circ F$

Namely for every triple of categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ there is a composition law

$$\circ : \mathrm{Ob}(\mathrm{Fun}(\mathcal{B},\mathcal{C})) \times \mathrm{Ob}(\mathrm{Fun}(\mathcal{A},\mathcal{B})) \longrightarrow \mathrm{Ob}(\mathrm{Fun}(\mathcal{A},\mathcal{C}))$$

coming from composition of functors. This composition law is associative, and identity functors act as units. In other words – forgetting about transformations of functors – we see that Cat forms a category. How does this structure interact with the morphisms between functors?

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow t"] & \mathcal{B} \arrow[r, "G"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, bend left, "G\circ F"] \arrow[r, bend right, "G\circ F'"'] & \mathcal{C} \end{tikzcd}$$

How to get the horizontal composition?

Well, given $t : F \to F'$ a transformation of functors $F, F' : \mathcal{A} \to \mathcal{B}$ and a functor $G : \mathcal{B} \to \mathcal{C}$ we can define a transformation of functors $G \circ F \to G \circ F'$. We will denote this transformation ${}_G t$. It is given by the formula $({}_G t)_x = G(t_x) : G(F(x)) \to G(F'(x))$ for all $x \in \mathcal{A}$. In this way composition with G becomes a functor

$$\overset{F}{\mathrm{Fun}(\mathcal{A},\mathcal{B})} \longrightarrow \overset{G\circ F}{\mathrm{Fun}(\mathcal{A},\mathcal{C})}.$$

To see this you just have to check that ${}_G(\mathrm{id}_F) = \mathrm{id}_{G\circ F}$ and that ${}_G(t_1 \circ t_2) = {}_G t_1 \circ {}_G t_2$. Of course we also have that ${}_{\mathrm{id}_\mathcal{A}} t = t$.

Similarly, given $s : G \to G'$ a transformation of functors $G, G' : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{A} \to \mathcal{B}$ a functor we can define s_F to be the transformation of functors $G \circ F \to G' \circ F$ given by $(s_F)_x = s_{F(x)} : G(F(x)) \to G'(F(x))$ for all $x \in \mathcal{A}$. In this way composition with F becomes a functor

$$\mathrm{Fun}(\mathcal{B},\mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{A},\mathcal{C}).$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow s"] & \mathcal{C} \end{tikzcd}$$

To see this you just have to check that $(\mathrm{id}_G)_F = \mathrm{id}_{G\circ F}$ and that $(s_1 \circ s_2)_F = s_{1,F} \circ s_{2,F}$. Of course we also have that $s_{\mathrm{id}_\mathcal{B}} = s$.

These constructions satisfy the additional properties

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow s"] & \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow s"] & \mathcal{C} \end{tikzcd}$$

$${}_{G_1}({}_{G_2} t) = {}_{G_1 \circ G_2} t, \ (s_{F_1})_{F_2} = s_{F_1 \circ F_2}, \text{ and } {}_H(s_F) = ({}_H s)_F$$

whenever these make sense. Finally, given functors $F, F' : \mathcal{A} \to \mathcal{B}$, and $G, G' : \mathcal{B} \to \mathcal{C}$ and transformations $t : F \to F'$, and $s : G \to G'$ the following diagram is commutative

$$\begin{tikzcd} G \circ F \arrow[r, "{}_G t"] \arrow[d, "s_F"'] & G \circ F' \arrow[d, "s_{F'}"] \\ G' \circ F \arrow[r, "{}_{G'} t"'] & G' \circ F' \end{tikzcd}$$

in other words ${}_{G'} t \circ s_F = s_{F'} \circ {}_G t$. To prove this we just consider what happens on any object $x \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{tikzcd}
G(F(x)) \arrow[r, "G(t_x)"] \arrow[d, "s_{F(x)}"'] & G(F'(x)) \arrow[d, "s_{F'(x)}"] \\
G'(F(x)) \arrow[r, "G'(t_x)"'] & G'(F'(x))
\end{tikzcd}$$

which is commutative because s is a transformation of functors. This compatibility relation allows us to define horizontal composition.

Then we have the conditions we need.

Definition 4.28.1. Given a diagram as in the left hand side of:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] & \mathcal{B} \arrow[r, bend left, "G"{name=U2}] \arrow[r, bend right, "G'"'{name=D2}] & \mathcal{C} & \text{gives} & \mathcal{A} \arrow[r, bend left, "G\circ F"{name=U3}] \arrow[r, bend right, "G'\circ F'"'{name=D3}] & \mathcal{C}
\arrow[Rightarrow, from=U, to=D, "t"]
\arrow[Rightarrow, from=U2, to=D2, "s"]
\arrow[Rightarrow, from=U3, to=D3, "s\star t"]
\end{tikzcd}$$

we define the *horizontal* composition $s \star t$ to be the transformation of functors ${}_{G'}t \circ s_F = s_{F'} \circ {}_G t$.

Now we define the 2-category.

Definition 4.29.1. A (strict) 2-*category* $\mathcal{C}$ consists of the following data

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.

(2) For each pair $x, y \in \mathrm{Ob}(\mathcal{C})$ a category $Mor_{\mathcal{C}}(x, y)$. The objects of $Mor_{\mathcal{C}}(x, y)$ will be called 1-*morphisms* and denoted $F : x \to y$. The morphisms between these 1-morphisms will be called 2-*morphisms* and denoted $t : F' \to F$. The composition of 2-morphisms in $Mor_{\mathcal{C}}(x, y)$ will be called *vertical* composition and will be denoted $t \circ t'$ for $t : F' \to F$ and $t' : F'' \to F'$.

(3) For each triple $x, y, z \in \mathrm{Ob}(\mathcal{C})$ a functor

$$(\circ, \star) : Mor_{\mathcal{C}}(y, z) \times Mor_{\mathcal{C}}(x, y) \longrightarrow Mor_{\mathcal{C}}(x, z).$$

The image of the pair of 1-morphisms (F, G) on the left hand side will be called the *composition* of F and G, and denoted $F \circ G$. The image of the pair of 2-morphisms (t, s) will be called the *horizontal* composition and denoted $t \star s$.

These data are to satisfy the following rules:

(1) The set of objects together with the set of 1-morphisms endowed with composition of 1-morphisms forms a category.

(2) Horizontal composition of 2-morphisms is associative.

(3) The identity 2-morphism $\mathrm{id}_{\mathrm{id}_x}$ of the identity 1-morphism id_x is a unit for horizontal composition.

Definition 4.29.2. Let C be a 2-category. A *sub* 2*-category* C' of C, is given by a subset $\mathrm{Ob}(C')$ of $\mathrm{Ob}(C)$ and sub categories $Mor_{C'}(x, y)$ of the categories $Mor_C(x, y)$ for all $x, y \in \mathrm{Ob}(C')$ such that these, together with the operations $\circ$ (composition 1-morphisms), $\circ$ (vertical composition 2-morphisms), and $\star$ (horizontal composition) form a 2-category.

All that we learned later will be some sub 2-category.

The notion of equivalence of categories that we defined in Section 4.2 extends to the more general setting of 2-categories as follows.

Definition 4.29.4. Two objects x, y $\in \mathrm{Ob}(Cat_2)$ of a 2-category are *equivalent* if there exist 1-morphisms $F : x \to y$ and $G : y \to x$ such that $F \circ G$ is 2-isomorphic to id_y and $G \circ F$ is 2-isomorphic to id_x.

Definition 4.30.1. A (strict) $(2, 1)$*-category* is a 2-category in which all 2-morphisms are isomorphisms.

Example 4.30.2. The 2-category *Cat*, see Remark 4.29.3, can be turned into a $(2, 1)$-category by only allowing isomorphisms of functors as 2-morphisms.

Ob same
morphism isomorphism

In fact, more generally any 2-category C produces a $(2, 1)$-category by considering the sub 2-category C' with the same objects and 1-morphisms but whose 2-morphisms are the invertible 2-morphisms of C. In this situation we will say "*let* C' *be the* $(2, 1)$*-category associated to* C" or similar. For example, the $(2, 1)$-category of groupoids means the 2-category whose objects are groupoids, whose 1-morphisms are functors and whose 2-morphisms are isomorphisms of functors. Except that this is a bad example as a transformation between functors between groupoids is automatically an isomorphism!

II. Categories over categories

Definition 4.32.1. Let C be a category. The *2-category of categories over* C is the 2-category defined as follows:

$$\begin{tikzcd} S \arrow[rr, bend left, "G"] \arrow[rr, bend right, "H"'] \arrow[dr, "p"'] & & S' \arrow[dl, "p'"] \\ & C & \end{tikzcd}$$

(1) Its objects will be functors $p : S \to C$.

(2) Its 1-morphisms $(S, p) \to (S', p')$ will be functors $G : S \to S'$ such that $p' \circ G = p$.

(3) Its 2-morphisms $t : G \to H$ for $G, H : (S, p) \to (S', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(S)$.

$$\begin{tikzcd} p'(G(x)) \to p'(H(x)) \arrow[d] \\ p(x) \to p(x) \end{tikzcd}$$

In this situation we will denote

$$Mor_{Cat/C}(S, S')$$

the category of 1-morphisms between (S, p) and (S', p')

$\mathcal{S} \to \mathcal{C}$

$$\begin{tikzcd} x \in \mathcal{S} \arrow[d] \\ U \in \mathcal{C} \end{tikzcd}$$

Definition 4.32.2. Let C be a category. Let $p : \mathcal{S} \to C$ be a category over C.

(1) The *fibre category* over an object $U \in \mathrm{Ob}(C)$ is the category $\mathcal{S}_U$ with objects

$$\mathrm{Ob}(\mathcal{S}_U) = \{x \in \mathrm{Ob}(\mathcal{S}) : p(x) = U\}$$

and morphisms

$$Mor_{\mathcal{S}_U}(x, y) = \{\phi \in Mor_{\mathcal{S}}(x, y) : p(\phi) = \mathrm{id}_U\}.$$

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "p"'] & y \arrow[d, "p"] \\ U \arrow[r, "\mathrm{id}_U"] & U \end{tikzcd}$$

(2) A *lift* of an object $U \in \mathrm{Ob}(C)$ is an object $x \in \mathrm{Ob}(\mathcal{S})$ such that $p(x) = U$, i.e., $x \in \mathrm{Ob}(\mathcal{S}_U)$. We will also sometime say that x *lies over* U.

(3) Similarly, a *lift* of a morphism $f : V \to U$ in C is a morphism $\phi : y \to x$ in $\mathcal{S}$ such that $p(\phi) = f$. We sometimes say that ϕ *lies over* f.

$$\begin{tikzcd} y \arrow[r] \arrow[d] & x \arrow[d] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

There are some observations we could make here. For example if $F : (\mathcal{S}, p) \to (\mathcal{S}', p')$ is a 1-morphism of categories over C, then F induces functors of fibre categories $F : \mathcal{S}_U \to \mathcal{S}'_U$. Similarly for 2-morphisms.

The following lemma is the 2-fibre products that we will meet for the stacks in groupoids

Lemma 4.32.3. *Let C be a category. The $(2, 1)$-category of categories over C has 2-fibre products. Suppose that $F : \mathcal{X} \to \mathcal{S}$ and $G : \mathcal{Y} \to \mathcal{S}$ are morphisms of categories over C. An explicit 2-fibre product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is given by the following description*

(1) *an object of $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ is a quadruple (U, x, y, f), where $U \in \mathrm{Ob}(C)$, $x \in \mathrm{Ob}(\mathcal{X}_U)$, $y \in \mathrm{Ob}(\mathcal{Y}_U)$, and $f : F(x) \to G(y)$ is an isomorphism in $\mathcal{S}_U$,*

(2) *a morphism $(U, x, y, f) \to (U', x', y', f')$ is given by a pair (a, b), where $a : x \to x'$ is a morphism in $\mathcal{X}$, and $b : y \to y'$ is a morphism in $\mathcal{Y}$ such that*

   (a) *a and b induce the same morphism $U \to U'$, and*

   (b) *the diagram*

$$\begin{tikzcd} F(x) \arrow[r, "f"] \arrow[d, "F(a)"] & G(y) \arrow[d, "G(b)"] \\ F(x') \arrow[r, "f'"] & G(y') \end{tikzcd}$$

   is commutative.

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "G"] \\ \mathcal{X} \arrow[r, "F"] & \mathcal{S} \end{tikzcd}$$

2-commute

$$\begin{tikzcd} x \arrow[r, "F"] \arrow[drr] & F(x) \arrow[r, phantom, "\cong"] \arrow[dr] & G(y) \arrow[d] & y \arrow[l, "G"'] \arrow[dl, bend left] \\ & & U & \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & x' \arrow[d] \\ U \arrow[r] & U' \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, dash] \arrow[d] & y' \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

The functors $p : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{S}} \mathcal{Y} \to \mathcal{Y}$ are the forgetful functors in this case. The transformation $\psi : F \circ p \to G \circ q$ is given on the object $\xi = (U, x, y, f)$ by $\psi_\xi = f : F(p(\xi)) = F(x) \to G(y) = G(q(\xi))$.

Ⅲ. Fibred categories

Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. Given an object $x \in \mathcal{S}$ with $p(x) = U$, and given a morphism $f : V \to U$, we can try to take some kind of "fibre product $V \times_U x$" (or a *base change* of x via $V \to U$). Namely, a morphism from an object $z \in \mathcal{S}$ into "$V \times_U x$" should be given by a pair (φ, g), where $\varphi : z \to x$, $g : p(z) \to V$ such that $p(\varphi) = f \circ g$. Pictorially:

$$\begin{tikzcd}
Mor_S(z,x) \arrow[dr, "p"'] & & Mor_S(p(z),p(z)) \arrow[dl, "p(\varphi)\circ"] \\
& Mor_C(p(z),p(x)) &
\end{tikzcd}$$

$$\begin{tikzcd}
z \arrow[r, dash] \arrow[d, squiggly, "p"] & ? \arrow[r] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
p(z) \arrow[r] & V \arrow[r, "f"] & U
\end{tikzcd}$$

$$\begin{tikzcd}
z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, squiggly] & & \\
& y \arrow[r, "\varphi"] \arrow[d, squiggly] & x \arrow[d, squiggly] \\
p(z) \arrow[r] & p(y) \arrow[r, "p(\varphi)"] & p(x)
\end{tikzcd}$$

If such a morphism $V \times_U x \to x$ exists then it is called a strongly cartesian morphism.

Definition 4.33.1. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. A *strongly cartesian morphism*, or more precisely a *strongly $\mathcal{C}$-cartesian morphism* is a morphism $\varphi : y \to x$ of $\mathcal{S}$ such that for every $z \in \mathrm{Ob}(\mathcal{S})$ the map

$$Mor_{\mathcal{S}}(z, y) \longrightarrow Mor_{\mathcal{S}}(z, x) \times_{Mor_{\mathcal{C}}(p(z),p(x))} Mor_{\mathcal{C}}(p(z), p(y)),$$

given by $\psi \longmapsto (\varphi \circ \psi, p(\psi))$ is bijective.

Note that by the Yoneda Lemma 4.3.5, given $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and the morphism $f : V \to U$ of $\mathcal{C}$, if there is a strongly cartesian morphism $\varphi : y \to x$ with $p(\varphi) = f$, then (y, φ) is unique up to unique isomorphism. This is clear from the definition above, as the functor

$$z \longmapsto Mor_{\mathcal{S}}(z, x) \times_{Mor_{\mathcal{C}}(p(z),U)} Mor_{\mathcal{C}}(p(z), V)$$

only depends on the data $(x, U, f : V \to U)$. Hence we will sometimes use $V \times_U x \to x$ or $f^*x \to x$ to denote a strongly cartesian morphism which is a lift of f.

Definition 4.33.5. Let $\mathcal{C}$ be a category. Let $p : \mathcal{S} \to \mathcal{C}$ be a category over $\mathcal{C}$. We say $\mathcal{S}$ is a *fibred category over $\mathcal{C}$* if given any $x \in \mathrm{Ob}(\mathcal{S})$ lying over $U \in \mathrm{Ob}(\mathcal{C})$ and any morphism $f : V \to U$ of $\mathcal{C}$, there exists a strongly cartesian morphism $f^*x \to x$ lying over f.

Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category. For every $f : V \to U$ and $x \in \mathrm{Ob}(\mathcal{S}_U)$ as in the definition we may choose a strongly cartesian morphism $f^*x \to x$ lying over f. By the axiom of choice we may choose $f^*x \to x$ for all $f : V \to U = p(x)$ simultaneously. We claim that for every morphism $\phi : x \to x'$ in $\mathcal{S}_U$ and $f : V \to U$ there is a unique morphism $f^*\phi : f^*x \to f^*x'$ in $\mathcal{S}_V$ such that

$$\begin{tikzcd}
f^*x \arrow[r, "f^*\phi"] \arrow[d] & f^*x' \arrow[d] \\
x \arrow[r, "\phi"] & x'
\end{tikzcd}$$

$$\begin{tikzcd}
f^*x \arrow[r] \arrow[d] & x \arrow[d] \\
V \arrow[r, "f"] & U
\end{tikzcd}$$

commutes. Namely, the arrow exists and is unique because $f^*x' \to x'$ is strongly cartesian. The uniqueness of this arrow guarantees that f^* (now also defined on morphisms) is a functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$.

$$\begin{tikzcd} f^*x \arrow[r] \arrow[d] & x \arrow[d] \\ V \arrow[r] & U \end{tikzcd}$$

Definition 4.33.6. Assume $p : \mathcal{S} \to \mathcal{C}$ is a fibred category.

(1) A *choice of pullbacks*[1] for $p : \mathcal{S} \to \mathcal{C}$ is given by a choice of a strongly cartesian morphism $f^*x \to x$ lying over f for any morphism $f : V \to U$ of $\mathcal{C}$ and any $x \in \mathrm{Ob}(\mathcal{S}_U)$.

(2) Given a choice of pullbacks, for any morphism $f : V \to U$ of $\mathcal{C}$ the functor $f^* : \mathcal{S}_U \to \mathcal{S}_V$ described above is called a *pullback functor* (associated to the choices $f^*x \to x$ made above).

Definition 4.33.9. Let $\mathcal{C}$ be a category. The 2-*category of fibred categories over* $\mathcal{C}$ is the sub 2-category of the 2-category of categories over $\mathcal{C}$ (see Definition 4.32.1) defined as follows:

(1) Its objects will be fibred categories $p : \mathcal{S} \to \mathcal{C}$.

(2) Its 1-morphisms $(\mathcal{S}, p) \to (\mathcal{S}', p')$ will be functors $G : \mathcal{S} \to \mathcal{S}'$ such that $p' \circ G = p$ and such that G maps strongly cartesian morphisms to strongly cartesian morphisms.

(3) Its 2-morphisms $t : G \to H$ for $G, H : (\mathcal{S}, p) \to (\mathcal{S}', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(\mathcal{S})$.

In this situation we will denote

$$Mor_{Fib/\mathcal{C}}(\mathcal{S}, \mathcal{S}')$$

the category of 1-morphisms between $(\mathcal{S}, p)$ and $(\mathcal{S}', p')$

Lemma 4.33.10. *Let $\mathcal{C}$ be a category. The $(2,1)$-category of fibred categories over $\mathcal{C}$ has 2-fibre products, and they are described as in Lemma 4.32.3.*

$$\begin{tikzcd} y \arrow[r, "\phi"] \arrow[d] & x \arrow[d] \\ V \arrow[r, "f"] & U \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, dashed, "\chi"] \arrow[rr, bend left, "\psi"] \arrow[d, squiggly, "p"'] & y \arrow[r, "\phi"] \arrow[d, squiggly] & x \arrow[d, squiggly] \\ p(z) \arrow[r, "f"] & p(y) \arrow[r, "p(\phi)"] & p(x) \end{tikzcd}$$

IV. Categories fibred in groupoids

Definition 4.35.1. Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. We say that $\mathcal{S}$ is *fibred in groupoids* over $\mathcal{C}$ if the following two conditions hold:

1. For every morphism $f : V \to U$ in $\mathcal{C}$ and every lift x of U there is a lift $\phi : y \to x$ of f with target x.
2. For every pair of morphisms $\phi : y \to x$ and $\psi : z \to x$ and any morphism $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : z \to y$ of f such that $\phi \circ \chi = \psi$.

Condition (2) phrased differently says that applying the functor p gives a bijection between the sets of dotted arrows in the following commutative diagram below:

$$\begin{tikzcd}
y \arrow[r] & x & p(y) \arrow[r] & p(x) \\
z \arrow[u, dashed] \arrow[ur] & & p(z) \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}$$

Another way to think about the second condition is the following. Suppose that $g : W \to V$ and $f : V \to U$ are morphisms in $\mathcal{C}$. Let $x \in \mathrm{Ob}(\mathcal{S}_U)$. By the first condition we can lift f to $\phi : y \to x$ and then we can lift g to $\psi : z \to y$. Instead of doing this two step process we can directly lift $g \circ f$ to $\gamma : z' \to x$. This gives the solid arrows in the diagram

(4.35.1.1)

$$\begin{tikzcd}
z' \arrow[rrd, "\gamma"] & & \\
z \arrow[u, dashed, <->] \arrow[r, "\psi"] \arrow[d, squiggly, "p"] & y \arrow[r, "\phi"] \arrow[d, squiggly, "p"] & x \arrow[d, squiggly, "p"] \\
W \arrow[r, "g"] & V \arrow[r, "f"] & U
\end{tikzcd}$$

where the squiggly arrows represent not morphisms but the functor p. Applying the second condition to the arrows $\phi \circ \psi, \gamma$ and id_W we conclude that there is a unique morphism $\chi : z \to z'$ in $\mathcal{S}_W$ such that $\gamma \circ \chi = \phi \circ \psi$. Similarly there is a unique morphism $z' \to z$. The uniqueness implies that the morphisms $z' \to z$ and $z \to z'$ are mutually inverse, in other words isomorphisms.

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & x \arrow[dl] \\
U &
\end{tikzcd}$$

$$\begin{tikzcd} z \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & y \arrow[d] \\ V \arrow[r] & U \arrow[r] & U \end{tikzcd}$$

Lemma 4.35.2. *Let $p : \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent*

(1) *$p : \mathcal{S} \to \mathcal{C}$ is a category fibred in groupoids, and*

(2) *all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$.*

Moreover, in this case every morphism of $\mathcal{S}$ is strongly cartesian. In addition, given $f^*x \to x$ lying over f for all $f : V \to U = p(x)$ the data $(U \mapsto \mathcal{S}_U, f \mapsto f^*, \alpha_{f,g}, \alpha_U)$ constructed in Lemma 4.33.7 defines a pseudo functor from $\mathcal{C}^{opp}$ in to the $(2,1)$-category of groupoids.

Proof. Assume $p : \mathcal{S} \to \mathcal{C}$ is fibred in groupoids. To show all fibre categories $\mathcal{S}_U$ for $U \in \mathrm{Ob}(\mathcal{C})$ are groupoids, we must exhibit for every $f : y \to x$ in $\mathcal{S}_U$ an inverse morphism. The diagram on the left (in $\mathcal{S}_U$) is mapped by p to the diagram on the right:

$f \circ g = id_x$

$$\begin{tikzcd} y \arrow[r, "f"] \arrow[rr, bend right] & x \arrow[rrr, bend right, "id_U"] & U \arrow[r, "\mathrm{id}_U"] & U \\ x \arrow[u, dashed, "g"] \arrow[ur, "\mathrm{id}_x"'] & & U \arrow[u, dashed] \arrow[ur, "\mathrm{id}_U"'] & \end{tikzcd}$$

Since only id_U makes the diagram on the right commute, there is a unique $g : x \to y$ making the diagram on the left commute, so $fg = \mathrm{id}_x$. By a similar argument there is a unique $h : y \to x$ so that $gh = \mathrm{id}_y$. Then $fgh = f : y \to x$. We have $fg = \mathrm{id}_x$, so $h = f$. Condition (2) of Definition 4.35.1 says exactly that every morphism of $\mathcal{S}$ is strongly cartesian. Hence condition (1) of Definition 4.35.1 implies that $\mathcal{S}$ is a fibred category over $\mathcal{C}$.

Conversely, assume all fibre categories are groupoids and $\mathcal{S}$ is a fibred category over $\mathcal{C}$. We have to check conditions (1) and (2) of Definition 4.35.1. The first condition follows trivially. Let $\phi : y \to x$, $\psi : z \to x$ and $f : p(z) \to p(y)$ such that $p(\phi) \circ f = p(\psi)$ be as in condition (2) of Definition 4.35.1. Write $U = p(x)$, $V = p(y)$, $W = p(z)$, $p(\phi) = g : V \to U$, $p(\psi) = h : W \to U$. Choose a strongly cartesian $g^*x \to x$ lying over g. Then we get a morphism $i : y \to g^*x$ in $\mathcal{S}_V$, which is therefore an isomorphism. We also get a morphism $j : z \to g^*x$ corresponding to the pair (ψ, f) as $g^*x \to x$ is strongly cartesian. Then one checks that $\chi = i^{-1} \circ j$ is a solution.

We have seen in the proof of (1) ⇒ (2) that every morphism of $\mathcal{S}$ is strongly cartesian. The final statement follows directly from Lemma 4.33.7. ☐

Definition 4.35.6. Let C be a category. The *2-category of categories fibred in groupoids over C* is the sub 2-category of the 2-category of fibred categories over C (see Definition 4.33.9) defined as follows:

1. Its objects will be categories $p : S \to C$ fibred in groupoids.
2. Its 1-morphisms $(S, p) \to (S', p')$ will be functors $G : S \to S'$ such that $p' \circ G = p$ (since every morphism is strongly cartesian G automatically preserves them).
3. Its 2-morphisms $t : G \to H$ for $G, H : (S, p) \to (S', p')$ will be morphisms of functors such that $p'(t_x) = \mathrm{id}_{p(x)}$ for all $x \in \mathrm{Ob}(S)$.

Note that every 2-morphism is automatically an isomorphism! Hence this is actually a $(2, 1)$-category and not just a 2-category. Here is the obligatory lemma on 2-fibre products.

V. Representable categories fibred in groupoids

Definition 4.39.1. Let us call a category a *setoid*[1] if it is a groupoid where every object has exactly one automorphism: the identity.

$$\begin{tikzcd} x \arrow[r] \arrow[dr] & y \arrow[d] \\ & U \end{tikzcd}$$

Definition 4.39.2. Let C be a category. A *category fibred in setoids* is a category fibred in groupoids all of whose fibre categories are setoids.

Definition 4.40.1. Let C be a category. A category fibred in groupoids $p : S \to C$ is called *representable* if there exists an object X of C and an equivalence $j : S \to C/X$ (in the 2-category of groupoids over C).

C/X : ob morphism $U \to X$ $V \overset{\circ}{\to} U \to X$

The usual abuse of notation is to say that X *represents* S and not mention the equivalence j. We spell out what this entails.

Lemma 4.40.2. *Let C be a category. Let $p : S \to C$ be a category fibred in groupoids.*

(1) *S is representable if and only if the following conditions are satisfied:*
  (a) *S is fibred in setoids, and*
  (b) *the presheaf $U \mapsto \mathrm{Ob}(S_U)/\cong$ is representable.*

(2) *If S is representable the pair (X, j), where j is the equivalence $j : S \to C/X$, is uniquely determined up to isomorphism.*

Proof. The first assertion follows immediately from Lemma 4.39.5. For the second, suppose that $j' : S \to C/X'$ is a second such pair. Choose a 1-morphism $t' : C/X' \to S$ such that $j' \circ t' \cong \mathrm{id}_{C/X'}$ and $t' \circ j' \cong \mathrm{id}_S$. Then $j \circ t' : C/X' \to C/X$ is an equivalence. Hence it is an isomorphism, see Lemma 4.38.6. Hence by the Yoneda Lemma 4.3.5 (via Example 4.38.7 for example) it is given by an isomorphism $X' \to X$. ☐

Lemma 4.40.3. *Let C be a category. Let $\mathcal{X}$, $\mathcal{Y}$ be categories fibred in groupoids over C. Assume that $\mathcal{X}$, $\mathcal{Y}$ are representable by objects X, Y of C. Then*

Deligne-Mumford Stacks

$$Mor_{Cat/C}(\mathcal{X}, \mathcal{Y}) \big/ \text{2-isomorphism} = Mor_C(X, Y)$$

⇐ Algebraic spaces

More precisely, given $\phi : X \to Y$ there exists a 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ which induces ϕ on isomorphism classes of objects and which is unique up to unique 2-isomorphism.

Proof. By Example 4.38.7 we have $C/X = S_{h_X}$ and $C/Y = S_{h_Y}$. By Lemma 4.39.6 we have

$$Mor_{Cat/C}(\mathcal{X}, \mathcal{Y}) \big/ \text{2-isomorphism} = Mor_{PSh(C)}(h_X, h_Y)$$

By the Yoneda Lemma 4.3.5 we have $Mor_{PSh(C)}(h_X, h_Y) = Mor_C(X, Y)$. ☐

$h_X = Mor_C(-, X)$ $V, U \in \mathcal{C}$

$\Updownarrow$

S_{h_X}

$Ob(S_{h_X}) = \{(U, x) \mid U \in Ob(\mathcal{C}),$
$x \in Ob(h_X(U))\}$

$(U,x), (V,y)$

$\nearrow$ $Mor_C((V,y), (U,x)) = \{f \in Mor_C(V,U) |$

hx $f^* x = y\}$

$p: \mathcal{C}/X \to \mathcal{C}$

$(\mathcal{C}/X)_U$ ob: $U \to X$

mor: id

Three ways of defining stacks

- pseudo-functor b/w 2-categories (Praphulla)
- category fibered in groupoids (Ruoxi)
- groupoid spaces

Recall: A 2-category $\mathcal{C}$ is given by the data:

- A class of objects $\boxed{ob(\mathcal{C})}$
- For each $X, Y \in Ob(\mathcal{C})$, a category $\boxed{Hom(X,Y}$

objects of Hom(X, Y) : 1-morphism

morphisms of Hom(X, Y) : 2-morphis

such that :

- compose 1-morphisms
- identity for objects
- vertical and horizontal composition of 2-morphisms and these are associative.
- identity for 1-morphism.
- horizontal and vertical compositions of 2-morphisms are compatible.

eg:
- Category of categories
- $\overset{U1}{\mathrm{Grpds}}$: category of groupoids
- Sch/S (Any usual category is 2-category)

2-morphisms are just identity morphis

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X \arrow[r, bend left=70, "f"{name=F}] \arrow[r, "f'"{description, name=F1}] \arrow[r, bend right=70, "f''"'{name=F2}] & Y \arrow[r, bend left=70, "g"{name=G}] \arrow[r, "g'"{description, name=G1}] \arrow[r, bend right=70, "g''"'{name=G2}] & Z
\arrow[Rightarrow, from=F, to=F1, "\alpha"]
\arrow[Rightarrow, from=F1, to=F2, "\alpha'"]
\arrow[Rightarrow, from=G, to=G1, "\beta"]
\arrow[Rightarrow, from=G1, to=G2, "\beta'"]
\end{tikzcd}$$

$$(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$$

$*$ ⇣ horizontal composition

A pseudo-functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ b/w 2-categories is given by:

1. $\forall\, X \in ob(\mathcal{C})$, $F(X) \in ob(\mathcal{D})$
2. $\forall\, X \xrightarrow{f} Y$ in $\mathcal{C}$, $F(X) \xrightarrow{\mathcal{F}(f)} F(Y)$ in $\mathcal{D}$.
3. $\forall$ 2-morphism $\alpha : f \Rightarrow g$ in $\mathcal{C}$, we have a 2-morphism $\mathcal{F}(\alpha) : \mathcal{F}(f) \Rightarrow \mathcal{F}(g)$.

s.t.

ⓐ $\mathcal{F}(Id_X) = Id_{\mathcal{F}(X)}$. ✓

ⓑ $\mathcal{F}(Id_f) = Id_{\mathcal{F}(f)}$ ✓

ⓒ For every diagram of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$

there is a 2-isomorphism

$$\boxed{\varepsilon_{g,f}} : \underline{\mathcal{F}(g)}\circ\underline{\mathcal{F}(f)} \Longrightarrow \underline{\mathcal{F}(g\circ f)}$$

$id_{\mathcal{F}(f)}$

s.t.

ⓘ $\varepsilon_{f,id_X} = \varepsilon_{id_Y,f} = id_{\mathcal{F}(f)}$

(ii) $\underline{\varepsilon}$ is associative, i.e.,

$$\begin{tikzcd}
\mathcal{F}(h)\circ\mathcal{F}(g)\circ\mathcal{F}(f) \arrow[r, Rightarrow, "\varepsilon_{h,g} * id_{\boxed{\mathcal{F}(f)}}"] \arrow[d, Rightarrow, "id_{\mathcal{F}(h)} * \boxed{\varepsilon_{g,f}}"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(h\circ g)\circ\mathcal{F}(f) \arrow[d, Rightarrow, "\boxed{\varepsilon_{h\circ g,f}}"] \\
\mathcal{F}(h)\circ\mathcal{F}(g\circ f) \arrow[r, Rightarrow, "\boxed{\varepsilon_{h,g\circ f}}"] & \mathcal{F}(h\circ g\circ f)
\end{tikzcd}$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

ⓓ Respects composition of 2-morphisms $\alpha : f \Rightarrow g$

$\mathcal{F}(\beta \circ \alpha) = \mathcal{F}(\beta) \circ \mathcal{F}(\alpha) \quad \beta : g \Rightarrow h$

ⓔ For every pair of 2-morphisms $\alpha : f \Rightarrow f'$, $\beta : g \Rightarrow g'$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}(g) \circ \mathcal{F}(f) \arrow[r, Rightarrow, "\mathcal{F}(\beta) * \mathcal{F}(\alpha)"] \arrow[d, Rightarrow, "\varepsilon_{g,f}"] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(g') \circ \mathcal{F}(f') \arrow[d, Rightarrow, "\varepsilon_{g',f'}"'] \\
\mathcal{F}(g \circ f) \arrow[r, Rightarrow, "\mathcal{F}(\beta * \alpha)"] & \mathcal{F}(g' \circ f')
\end{tikzcd}$$

Defn: Let $\underline{C}$ be a category. A **prestack** $\mathcal{X}$ is a pseudo-functor

$$\mathcal{X} : \boxed{C^{op}} \longrightarrow \boxed{\mathrm{Grpds}} .$$

→ 2-category

This is the same as:

1. $\forall\ \underline{X \in ob(C)}$, an object $\mathcal{X}(X)$ in Grpds.
2. $\forall\ X \xrightarrow{f} Y$ in C, a functor
$$\underline{f^* = \mathcal{X}(f)} : \mathcal{X}(Y) \longrightarrow \mathcal{X}(X)$$
3. For each diagram in C of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$

we have an invertible natural

transformation in Grpds

$$\boxed{\varepsilon_{g,f}} : \underline{(g\circ f)^* \Rightarrow f^*\circ g^*}$$

s.t.

→ fill these arrows

$$\begin{tikzcd}
(h\circ g\circ f)^* \arrow[r, Rightarrow, "?"] \arrow[d, Rightarrow, "?"'] \arrow[dr, phantom, "\circlearrowright"] & (g\circ f)^*\circ h^* \arrow[d, Rightarrow, "?"'] \\
f^*\circ(h\circ g)^* \arrow[r, Rightarrow, "?"] & f^*\circ g^*\circ h^*
\end{tikzcd}$$

Stacks

Let C_τ be a site. A stack $\mathcal{X}$ is a prestack satisfying:

Let $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ be a covering family, then

1. (Glue objects) Given objects $x_i \in \mathcal{X}(U_i)$ and morphisms $\varphi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ satisfying the cocycle conditions

$$U_{ij} = U_i \times_U U_j$$

Descent Datum

$$\varphi_{ij}|_{U_{ijk}} \circ \varphi_{jk}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}},$$

then $\exists$ an object $x \in \mathcal{X}(U)$ and an isomorphism

$$\varphi_i : x|_{U_i} \xrightarrow{\sim} x_i \quad \forall i$$

s.t. $$\varphi_{ji} \circ \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}} .$$

2. (Glue morphisms uniquely)

Given objects x and y of $\mathcal{X}(U)$ and morphisms $\varphi_i : x|_{U_i} \longrightarrow y|_{U_i}$

s.t.

$$\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}} ,$$

then $\exists!$ morphism $\eta : x \longrightarrow y$

s.t. $\eta|_{U_i} = \varphi_i$.

More concisely:

Prpn: Let C_τ be a site. A stack $\mathcal{X}$ is a prestack satisfying:

glue objects

① every descent datum is effective

② $\forall\, U \in ob(C)$ and $\forall\, x, y \in \mathcal{X}(U)$, the presheaf of Sets

$$\boxed{\underline{\mathrm{Isom}}_U(x,y)} : (C/U)^{op} \longrightarrow (\underline{Sets})$$

$$(U' \xrightarrow{f} U) \mapsto \underset{\boxed{\mathcal{X}(U)}}{\mathrm{Hom}(f^*x, f^*y)}$$

$$\begin{tikzcd} f^*\varepsilon_1, f^*\varepsilon_2 \arrow[d] & \varepsilon_1, \varepsilon_2 \arrow[d] \\ U' \arrow[r, "f"] & U \end{tikzcd}$$

is a sheaf on the site $(C/U)_\tau$.

Examples of Stacks

① For any site C_τ, any sheaf

$$\mathcal{F} : C^{op} \longrightarrow (\text{Sets})$$

gives a stack.

Thus for $(Sch/S)_\tau$, we get

$$\boxed{\text{S-schemes} \hookrightarrow \text{Sheaves} \hookrightarrow \text{Stacks on } (Sch/S)_\tau}$$

Any set is a groupoid (only morphisms are identity morphisms)

② (Moduli stack of quasi-coherent sheaves on a scheme)

Let X be an S-scheme and consider the site $(Sch/S)_\tau$

$$\mathcal{Q}Coh_X : (Sch/_S) \longrightarrow \underline{Grpds}$$

$$\begin{tikzcd} \boxed{z} \arrow[d, mapsto] & \in & \overbrace{X \times U}^{\mathcal{E}} \arrow[d, "\pi"] \\ u & \in & U \end{tikzcd}$$

$$\underline{U \longmapsto \mathcal{Q}Coh_X(U)}$$

$\mathcal{E}$ is $\mathcal{O}_{X\times U}$-module!!

$\{$quasi-coherent $\mathcal{O}_{X\times U}$-modules $\underline{\text{flat}}$ over $U\}$ + $\underline{\text{isomorphisms}}$

$$(U' \xrightarrow{f} U) \mapsto f^* : \mathcal{Q}Coh_X(U) \to \mathcal{Q}Coh_X(U')$$

$$f^*(\mathcal{E}) := (Id_X \times f)^*(\mathcal{E}) \leftarrow \mathcal{E}$$

$$\underbrace{X \times_S U'} \xrightarrow{\boxed{Id_X \times f}} \overbrace{X \times_S U}^{\mathcal{E}}$$

$\boxed{\text{2-isomorphisms}}$: $(g^* \circ f^*)(\mathcal{E}) \cong (f \circ g)^*(\mathcal{E})$

$$\mathcal{O}_{U,u} \to \mathcal{O}_{X\times U, z} \swarrow \mathcal{E}_z \Rightarrow \mathcal{E}_z \text{ as } \mathcal{O}_{U,u}\text{-module is flat}$$

If S is locally noetherian and X is locally of finite type over S, then Coh_X (stack of coherent sheaves on X).

③ (Moduli stack of vector bundles over a scheme): $X = S$-scheme

$$\mathrm{Bun}^n_X : (Sch/S)^{op} \longrightarrow \mathrm{Grpds}$$

is a stack in fpqc ($\Rightarrow$ fppf, étale, smooth, Zariski)

$$\begin{tikzcd} \overset{\rotatebox{90}{$\in$}}{U} \arrow[r, mapsto] & \mathrm{Bun}^n_X(U) \arrow[d, equal] \\ & \{\text{vector bundles } \mathcal{E} \text{ on } X \times U\} + \text{iso.} \end{tikzcd}$$

f^* given by pullback of vector bundles.

$\left(Sch/k\right)_{\Sigma}$, X = smooth proj. irreducible alg. curve $/k$ of genus g.

- $Bun_X^{n,d}$ = v.b. w/ each fiber having degree d
- $Bun_X^{ss,n,d}$ = in addition, each fiber is a s.s. v.b. $u \in U$
- $Bun_X^{st,n,d}$

$$\begin{tikzcd} \text{v.b.} \arrow[r] & \mathcal{E}_u \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ & \underline{X \times \{u\}} \arrow[r] & X \times U \end{tikzcd}$$

- $Bun_{G,X}$

G = reductive alg. gp. $/k$

eg: GL_n, $SO(n)$, Sp_{2n}, …

→ A vector bundle E is semi-stable if $\forall$ $F \subsetneq E$ sub-bundles, $\mu(F) \leq \mu(E$

④ (Quotient stack) : (Sch/S)

X = noetherian S-scheme $\longrightarrow$ variety

G = affine smooth gp. S-scheme

$$\rho : X \times G \longrightarrow X$$

action of G on X

$$[X/G] : (Sch/S)^{op} \longrightarrow Grpds$$

$$U \mapsto [X/G](U)$$

$$[X/G](U) = \overbrace{\left\{ \begin{tikzcd} \mathcal{E} \arrow[d, "G\text{-bundle}"] \\ U \end{tikzcd} + \mathcal{E} \xrightarrow[\alpha]{G\text{-eq.}} X \right\}}$$

+ morphisms are isomorphisms of G-bundles commuting w/ G-equiv. morphism

f^* = given by pullback of G-bundles

$$[X/G] = \text{stack on } (Sch/S)_{ét}$$

- $S = \operatorname{Spec} k$

  $X = \operatorname{Spec} k = *$

  G acts trivially on X

Then $[*/G]$ gives precisely

G-bundles on $U \in (Sch/k)$.

$\longrightarrow \boxed{BG}$ = classifying stack.

$$[*/G]\Big(\underset{\substack{\downarrow \\ k\text{-scheme}}}{U}\Big) = \left\{ \begin{array}{c} E \\ \downarrow G \\ U \end{array} \right\}$$

⑤ (Moduli stack of algebraic curves)

(Sch/S), $g \geq 2$

$$\mathcal{M}_g : (Sch/S)^{op} \longrightarrow Grpds$$

$$U \longmapsto \mathcal{M}_g(U)$$

$$\parallel$$

$\{$family of alg. curves over $U\}$ + isomorphisms of such families over U

A proper and flat morphism

$$\begin{tikzcd} C_u \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ u \arrow[r] & U \end{tikzcd}$$

whose fibers are reduced, connected, 1-dim'l schemes C_u w/ genus g $(= \dim H^1(C_u, \mathcal{O}_{C_u}))$

f^* given by pullback, i.e.:

$$U' \xrightarrow{f} U$$

$$\begin{tikzcd} U' \times_U C \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ U' \arrow[r, "f"] & U \end{tikzcd}$$

$\mathcal{M}_g$ is a stack in **the** étale topology.

- Later, $\mathcal{M}_g$ is actually a quotient' stack.

⑥ (Moduli stack of stable algebraic curves) (Stack on big étale site)

$g \geq 2$:

$$\underbrace{\overline{M}_g}_{\text{proper}} : (Sch/S)^{op} \longrightarrow Grpds$$

$$U \longmapsto \widetilde{M}_g(U)$$

$$\shortparallel$$

$\{\underbrace{\text{family of stable alg. curves of genus } g}_{\text{family of alg. curves s.t.}}\}$ + isomorphisms of such family over U

family of alg. curves s.t.

① the only singularities of C_u are ordinary double points.

② If D is a non-singular rational component of C_u, then D meets the other components of C_u in more than 2 points.

PRELIMINARY TO ALGEBRAIC STACKS

Last time. Three ways of defining stacks

- pseudo-functors (Praphulla and Rahul)
- category fibred in groupoids (me)
- groupoid spaces

{ Example
{ Why ?

0 { Introduction to Algebraic Stacks

- Representable by { Schemes / Algebraic Spaces
- Maps representable by Algebraic Spaces
- Property P

Reference. Stacks Project Chapter 92, 103

Intro to Algebraic Stacks

simple language ⟶ moduli problem $\begin{cases} \text{local} \\ \text{global} \end{cases}$

Def. S scheme

Elliptic curve over S is a triple (E, f, O) where $\begin{cases} E \text{ scheme} \\ f: E \to S \text{ morphism of schemes} \\ O: S \to E \end{cases}$

$$\begin{tikzcd} E \arrow[d] \arrow[d, bend left, "f"] \\ S \arrow[u, bend left, "O"] \end{tikzcd}$$

S.t.

(1) $f: E \to S$ proper, smooth of relative dimension 1

(2) $\forall s \in S$ the fibre E_s is a connected curve of genus 1

i.e. $\underbrace{H^0(E_s, \mathcal{O})}_{\text{connected}}, \underbrace{H^1(E_s, \mathcal{O})}_{\text{genus } 1}$ are 1-dim $k(s)$ vector space

(3) O is a section of f

Morphism $(E, f, O)/S$ and $a: S \to S'$

$\alpha: E \to E'$

S.t. the diagram commutes and inner square cartesian

$$\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[d, "f"] & E' \arrow[d, "f'"'] \\
S \arrow[r, "a"] \arrow[u, bend left, "0"] & S' \arrow[u, bend right, "0'"']
\end{tikzcd}$$

Want to define the stack of elliptic curves $\mathcal{M}_{1,1}$

i.e. a category endowed with a functor

$$p: \mathcal{M}_{1,1} \to Sch$$
$$(E,f,0)/S \mapsto S$$

Enlarge the category Sch

(1) start with Sch $\quad S \mapsto (E,f,0)/S$

(2) add $\mathcal{M}_{1,1}$

(3) morphism $S \to \mathcal{M}_{1,1}$ is an elliptic curve $(E,f,0)/S$

(4) diagram

$$\begin{tikzcd}
S \arrow[rr, "a"] \arrow[dr, "{(E,f,0)}"'] & & S' \arrow[dl, "{(E',f',0')}"] \\
& \mathcal{M}_{1,1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\alpha"] \arrow[d] & E' \arrow[d] \\
S \arrow[r, "a"] & S'
\end{tikzcd}$$

commutative $\Longleftrightarrow$ $\exists \ \alpha: E \to E' / a$

(5) diagram glue

$$\begin{tikzcd}
S \arrow[r, "a"'] \arrow[dr] & S' \arrow[r, "a'"'] \arrow[d] & S'' \arrow[dl] \\
& M_{1,1} &
\end{tikzcd}$$

(6) $S \xrightarrow{a} S' \xrightarrow{(E',f',0')} M_{1,1}$

is given by $(E' \times_{S'} S, f' \times_{S'} S, 0' \times_{S'} S)$

What else?

① morphism $F: M_{1,1} \to T$ is a rule

s.t.
$$\begin{tikzcd}
S \arrow[r, "a"'] \arrow[dr, "{(E,f,0)}"'] & S' \arrow[d, "{(E',f',0')}"] \\
& M_{1,1}
\end{tikzcd}$$
commutative

$\Downarrow$

$$\begin{tikzcd}
S \arrow[r, "a"'] \arrow[dr, "{F(E,f,0)}"'] & S' \arrow[d, "{F(E',f',0')}"] \\
& T
\end{tikzcd}$$
commutative

② morphism: $M_{1,1} \to M_{1,1}$

consist of identity for now

Is this well defined?

Yes! With the language of 2-categories

Fibre products

$$\begin{tikzcd}
& & S \arrow[dr, "{(E,f,0)}"] & \\
T \arrow[urr, dashed, "a"] \arrow[r, dashed] \arrow[drr, dashed, "a'"'] & ? \arrow[ur, dashed] \arrow[dr, dashed] & & M_{1,1} \\
& & S' \arrow[ur, "{(E',f',0')}"'] &
\end{tikzcd}$$

morphism $T \to ?$ should be a triple (a, a', α)

$a : T \to S$

$a' : T \to S'$

$\alpha : E \times_{S,a} T \to E' \times_{S',a'} T$

isomorphism of elliptic curves $/T$

Key Fact. The functor $Sch^{op} \to Sets$

$T \mapsto \{(a, a', \alpha) \text{ above}\}$

is representable by a scheme $S \times_{\mathcal{M}_{1,1}} S'$

★ **Def.** $S \to \mathcal{M}_{1,1}$ is smooth if $\forall\, S' \to \mathcal{M}_{1,1}$ the projection morphism

$S \times_{\mathcal{M}_{1,1}} S' \to S'$

is smooth.

(compactible with morphism of schemes)

Def. $\mathcal{M}_{1,1}$ is an algebraic stack if and only if

(1) We have descent for objects for the étale topology on Sch. (gluing)

(2) The key fact holds.

(3) there exists $S \to \mathcal{M}_{1,1}$ surjective and smooth.

Finally. A smooth cover

(in some sense existence ⇒ key fact)

use Weierstrass equation

$W = \mathrm{Spec}(\mathbb{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta])$

$\Delta \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ polynomial

$\mathbb{P}^2_W \supset E_W : \; zy^2 + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3$

$f_W : E_W \to W$ projection

$O_W : W \to E_W$ the section given by $(0:1:0)$

Lemma. The morphism $\underset{\text{scheme}}{W} \xrightarrow{(E_W, f_W, O_W)} \mathcal{M}_{1,1}$ is smooth and surjective

Proof. ① Surjective. By the fact that every elliptic curve over a field has a Weierstrass equation

② Smooth.

consider sub group scheme

$$H = \left\{ \begin{pmatrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix} \middle| \begin{array}{l} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{array} \right\} \subset GL_{3,\mathbb{Z}}$$

action : $H \times W \to W$

result : (1) any $(E, f, O)/S$ has Zariski locally on S a Weierstrass equation

(2) any two differ (Zariski locally) by H

$\Rightarrow$ $W \to \mathcal{M}_{1,1}$ is an H-torsor

Since $H \to \mathrm{Spec}(\mathbb{Z})$ smooth

$\Rightarrow$ torsors / smooth group schemes are smooth

$\square$

Remark. Also shows that $\mathcal{M}_{1,1} = [W/H]$ is a global quotient stack.

(half cases)

What helps?

$W \to \mathcal{M}_{1,1}$ surj./ smooth

Local

$\mathcal{M}_{1,1} \to \mathrm{Spec}(\mathbb{Z})$ smooth

$\Leftrightarrow$ $W \to \mathrm{Spec}(\mathbb{Z})$ smooth

Global

$\mathcal{M}_{1,1}$ quasi-compact $\Leftarrow$ W quasi-compact

$\mathcal{M}_{1,1}$ irreducible $\Leftarrow$ W irreducible

Quasi-coherent sheaves

$QCoh(\mathcal{M}_{1,1}) = H$- equivariant quasi-coherent modules on W

Picard group

$\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathrm{Pic}_H(W) = \mathbb{Z}/12\mathbb{Z}$

NEXT : Representable

Recall. Site

Definition 58.10.2. A *site*[1] consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target called *coverings*, such that

(1) (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,

(2) (locality) if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$, then

$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j)\in\prod_{i\in I}\{i\}\times I_i}$$

is also a covering, and

(3) (base change) if $\{U_i \to U\}_{i\in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then

   (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$, and
   (b) $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

fppf covering

Definition 34.7.1. Let T be a scheme. An *fppf covering of* T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ of schemes such that each f_i is flat, locally of finite presentation and such that $T = \bigcup f_i(T_i)$.

Representable by schemes

Let $\mathcal{X}$ be a category fibred in groupoids over $(Sch/S)_{fppf}$. Recall that $\mathcal{X}$ is said to be *representable* if there exists a scheme $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and an equivalence

$$j : \mathcal{X} \longrightarrow (Sch/U)_{fppf}$$

of categories over $(Sch/S)_{fppf}$, see Categories, Definition 4.40.1. We will sometimes say that $\mathcal{X}$ is *representable by a scheme* to distinguish from the case where $\mathcal{X}$ is representable by an algebraic space (see below).

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by U, V, then we have

$$Mor_{Cat/(Sch/S)_{fppf}}(\mathcal{X},\mathcal{Y}) \Big/ \text{2-isomorphism} = Mor_{Sch/S}(U, V) \qquad (92.4.0.1)$$

{① F sheaf
② similar to algebraic stacks

Algebraic spaces ⇒ Algebraic stacks

Example 4.37.1. This example is the analogue of Example 4.36.1, for "presheaves of groupoids" instead of "presheaves of categories". The output will be a category fibred in groupoids instead of a fibred category. Suppose that $F : \mathcal{C}^{opp} \to \textit{Groupoids}$ is a functor to the category of groupoids, see Definition 4.29.5. For $f : V \to U$ in $\mathcal{C}$ we will suggestively write $F(f) = f^*$ for the functor from $F(U)$ to $F(V)$. We construct a category $\mathcal{S}_F$ fibred in groupoids over $\mathcal{C}$ as follows. Define

$$\mathrm{Ob}(\mathcal{S}_F) = \{(U, x) \mid U \in \mathrm{Ob}(\mathcal{C}), x \in \mathrm{Ob}(F(U))\}.$$

For $(U, x), (V, y) \in \mathrm{Ob}(\mathcal{S}_F)$ we define

$$\begin{aligned} Mor_{\mathcal{S}_F}((V, y), (U, x)) &= \{(f, \phi) \mid f \in Mor_{\mathcal{C}}(V, U), \phi \in Mor_{F(V)}(y, f^*x)\} \\ &= \coprod\nolimits_{f \in Mor_{\mathcal{C}}(V,U)} Mor_{F(V)}(y, f^*x) \end{aligned}$$

$$\begin{tikzcd} & V \arrow[r, "f"] \arrow[dl] \arrow[d] & U \arrow[d] \\ y \arrow[r, "\phi"] & f^*x \arrow[r] & x \end{tikzcd}$$

In order to define composition we use that $g^* \circ f^* = (f \circ g)^*$ for a pair of composable morphisms

Representable by Algebraic Spaces

Definition 92.8.1. Let S be a scheme contained in Sch_{fppf}. A category fibred in groupoids $p : \mathcal{X} \to (Sch/S)_{fppf}$ is called *representable by an algebraic space over* S if there exists an algebraic space F over S and an equivalence $j : \mathcal{X} \to \mathcal{S}_F$ of categories over $(Sch/S)_{fppf}$.

If $\mathcal{X}, \mathcal{Y}$ are fibred in groupoids and representable by algebraic spaces F, G over S, then we have

$$Mor_{Cat/(Sch/S)_{fppf}}(\mathcal{X}, \mathcal{Y}) \Big/ \text{2-isomorphism} = Mor_{Sch/S}(F, G) \quad (92.8.2.1)$$

see Categories, Lemma 4.39.6. More precisely, any 1-morphism $\mathcal{X} \to \mathcal{Y}$ gives rise to a morphism $F \to G$. Conversely, give a morphism of sheaves $F \to G$ over S there exists a 1-morphism $\phi : \mathcal{X} \to \mathcal{Y}$ which gives rise to $F \to G$ and which is unique up to unique 2-isomorphism.

Map representable by Algebraic Spaces

Definition 92.9.1. Let S be a scheme contained in Sch_{fppf}. A 1-morphism $f : \mathcal{X} \to \mathcal{Y}$ of categories fibred in groupoids over $(Sch/S)_{fppf}$ is called *representable by algebraic spaces* if for any $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y : (Sch/U)_{fppf} \to \mathcal{Y}$ the category fibred in groupoids

$$(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$$

over $(Sch/U)_{fppf}$ is representable by an algebraic space over U.

Choose an algebraic space F_y over U which represents $(Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X}$. We may think of F_y as an algebraic space over S which comes equipped with a canonical morphism $f_y : F_y \to U$ over S, see Spaces, Section 63.16. Here is the diagram

(92.9.1.1)
$$\begin{tikzcd}
F_y \arrow[d, "f_y"'] & (Sch/U)_{fppf} \times_{y,\mathcal{Y}} \mathcal{X} \arrow[l, squiggly] \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & \mathcal{X} \arrow[d, "f"] \\
U & (Sch/U)_{fppf} \arrow[l, squiggly] \arrow[r, "y"] & \mathcal{Y}
\end{tikzcd}$$

where the squiggly arrows represent the construction which associates to a stack fibred in setoids its associated sheaf of isomorphism classes of objects. The right square is 2-commutative, and is a 2-fibre product square.

Properties.

① object ⇒ morphism

Lemma 92.9.4. *Let S be an object of Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over S. If $\mathcal{X}$ and $\mathcal{Y}$ are representable by algebraic spaces over S, then the 1-morphism f is representable by algebraic spaces.*

② base change

Lemma 92.9.7. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the fibre product diagram*

$$\begin{tikzcd}
\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"'] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{Z} \arrow[r, "g"] & \mathcal{Y}
\end{tikzcd}$$

Then the base change f' is a 1-morphism representable by algebraic spaces.

③ fibre product

Lemma 92.9.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$ Let $f : \mathcal{X} \to \mathcal{Y}, g : \mathcal{Z} \to \mathcal{Y}$ be 1-morphisms. Assume*

1. *f is representable by algebraic spaces, and*
2. *$\mathcal{Z}$ is representable by an algebraic space over S.*

Then the 2-fibre product $\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space.

④ composition

Lemma 92.9.9. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. If $f : \mathcal{X} \to \mathcal{Y}, g : \mathcal{Y} \to \mathcal{Z}$ are 1-morphisms representable by algebraic spaces, then*

$$g \circ f : \mathcal{X} \longrightarrow \mathcal{Z}$$

is a 1-morphism representable by algebraic spaces.

⑤ product

Lemma 92.9.10. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. Then*

$$f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$$

is a 1-morphism representable by algebraic spaces.

Property P

Definition 92.10.1. Let S be a scheme contained in Sch_{fppf}. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of categories fibred in groupoids over $(Sch/S)_{fppf}$. Assume f is representable by algebraic spaces. Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which

1. is preserved under any base change, and
2. is fppf local on the base, see Descent on Spaces, Definition 72.9.1.

In this case we say that f has *property $\mathcal{P}$* if for every $U \in \mathrm{Ob}((Sch/S)_{fppf})$ and any $y \in \mathcal{Y}_U$ the resulting morphism of algebraic spaces $f_y : F_y \to U$, see diagram (92.9.1.1), has property $\mathcal{P}$.

① base change

Lemma 92.10.6. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 92.10.1. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism representable by algebraic spaces. Let $g : \mathcal{Z} \to \mathcal{Y}$ be any 1-morphism. Consider the 2-fibre product diagram*

$$\begin{tikzcd}
\mathcal{Z} \times_{g,\mathcal{Y},f} \mathcal{X} \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{Z} \arrow[r, "g"] & \mathcal{Y}
\end{tikzcd}$$

If f has $\mathcal{P}$, then the base change f' has $\mathcal{P}$.

② composition

Lemma 92.10.5. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be categories fibred in groupoids over $(Sch/S)_{fppf}$. Let $\mathcal{P}$ be a property as in Definition 92.10.1 which is stable under composition. Let $f : \mathcal{X} \to \mathcal{Y}$, $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms which are representable by algebraic spaces. If f and g have property $\mathcal{P}$ so does $g \circ f : \mathcal{X} \to \mathcal{Z}$.*

③ product

Lemma 92.10.8. *Let S be a scheme contained in Sch_{fppf}. Let $\mathcal{P}$ be a property as in Definition 92.10.1 which is stable under composition. Let $\mathcal{X}_i, \mathcal{Y}_i$ be categories fibred in groupoids over $(Sch/S)_{fppf}$, $i = 1, 2$. Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$, $i = 1, 2$ be 1-morphisms representable by algebraic spaces. If f_1 and f_2 have property $\mathcal{P}$ so does $f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$.*

Recall: $(Sch/S)_{\acute{e}t}$ = big etale site

Artin Stack: A stack $\mathcal{X}$ over the site $(Sch/S)_{\acute{e}t}$ such that:

1. $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ is representable by alg. spaces and quasi-compact.
2. $\exists$ a scheme X, called an atlas and a surjective smooth morphism $X \longrightarrow \mathcal{X}$.

$$\begin{tikzcd}
\text{alg. space} & \mathcal{X} \times_{\mathcal{Y}} X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \mathcal{X} \arrow[r] & \mathcal{Y}
\end{tikzcd}$$

Deligne - Mumford stack: A stack $\mathcal{X}$ over $(Sch/S)_{ét}$ such that

1. $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable by schemes, quasi-compact and separated.
2. $\exists$ a scheme X, called an atlas and a surjective etale morphism $X \rightarrow \mathcal{X}$.

$$\boxed{\text{DM-stacks} \subseteq \text{Artin stacks.}}$$

Prpn: $\mathcal{X}$ Artin : Δ is of finite type.

$\mathcal{X}$ DM : Δ is unramified.

$\boxed{\text{Cor:}}$ Let $\mathcal{X}$ be a DM stack and X be a quasi-compact scheme. If $x \in ob(\mathcal{X}(X))$, then x has only finitely many automorphisms.

$\boxed{\mathcal{X}/\mathrm{Spec}\, k}$

X Bun_n $\quad x_0 \in ob(\mathcal{X}(k))$

Prpn: Let $\mathcal{X}$ be an Artin stack.

TFAE: $\quad x_0' \in \mathcal{X}\left(\mathrm{Spec}\, \frac{k[\varepsilon]}{(\varepsilon^2)}\right)$ choosing a tangent vector at x_0

① $\mathcal{X}$ is a DM stack.

② Δ is unramified.

③ No object has non-trivial infinitesimal automorphisms.

$$\begin{tikzcd} \mathrm{Aut}(x_0') \arrow[d] \\ \mathrm{Aut}(x_0) \end{tikzcd}$$

$$\boxed{\mathrm{Inf}_{x_0}\mathcal{X}} = \mathrm{Ker}\left(\mathrm{Aut}_{\frac{k[\varepsilon]}{(\varepsilon^2)}}(x_0') \to \mathrm{Aut}_k(x_0)\right)$$

Defn: An Artin stack $\mathcal{X}$ is called smooth (resp. reduced, resp. locally noetherian, resp. normal, resp. regular) if $\exists$ an atlas $x : X \longrightarrow \mathcal{X}$ with the scheme X being smooth (resp. reduced, resp. locally noetherian, resp. normal, resp. regular).

Defn: Let P be a property of morphisms of schemes $f : X \to Y$ such that f has P iff for some smooth surjective morphism $Y' \longrightarrow Y$ the induced morphism $f' : X \times_Y Y' \to Y'$ has P.

A representable morphism $F : \mathcal{X} \longrightarrow \mathcal{Y}$ of algebraic stacks has P iff for some atlas $y : Y \longrightarrow \mathcal{Y}$ the induced morphism

$$\overline{F} : Y \times_{\mathcal{Y}} \mathcal{X} \longrightarrow Y$$

of schemes has P.

eg: closed / open embedding, affine, finite, proper etc.

$\rightsquigarrow$ Get notions of open and closed substacks.

$\mathcal{X} \xhookrightarrow[\text{embedding}]{\text{open}} \mathcal{Y} \implies \mathcal{X}$ is an open substack of $\mathcal{Y}$

Connected algebraic stack:

Not isomorphic to the disjoint union of two non-empty closed (or open) algebraic substacks.

Irreducible algebraic stack: not the union of two non-empty closed substacks.

Prpn: A locally noetherian alg. stack is in one and only one way the disjoint union of connected alg. stacks, called the connected components.

Prpn (Lifting criterion for smooth morphisms) A morphism of schemes $f: X \to Y$ is smooth iff f is locally of finite presentation and for all local Artin algebras A with ideal $I \subset A$ with $I^2 = (0)$ there is a lifting

$$\begin{tikzcd}
\operatorname{Spec}(A/I) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\operatorname{Spec}(A) \arrow[r] \arrow[ur, dashed, "?"] & Y
\end{tikzcd}$$

$A = \frac{k[\varepsilon]}{(\varepsilon^2)}$

$I = (\bar{\varepsilon})$

$$\begin{tikzcd}
\operatorname{Spec}(k) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) \arrow[r] \arrow[ur, dashed, "?"] & Y
\end{tikzcd}$$

$\ni (x, \bar{v})$ ← tangent vector at x

$\mapsto$

$(f(x), v)$ ← tangent vector at f(x

Prpn: Let $\mathcal{X}$ be an algebraic stack which is locally of finite presentation over Spec(k) such that the structure morphism $\mathcal{X} \longrightarrow \mathrm{Spec}(k)$ satisfies the lifting criterion for smoothness. Then $\mathcal{X}$ is smooth.

Proof: Recall: $\mathcal{X}$ is smooth iff $\exists$ an atlas $x : X \longrightarrow \mathcal{X}$ such that X is smooth.

Take any atlas $x : X \longrightarrow \mathcal{X}$ and we would like to show $X \longrightarrow \mathrm{Spec}\, k$ is smooth.

We use the lifting criterion

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "?"] & \mathrm{Spec}(k)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r, "f"] \arrow[dr, "x \circ f"'] \arrow[dd, hook] & X \arrow[d, "x"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathcal{X} \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}\, k
\end{tikzcd}$$

→ satisfy the lifting criteri

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(A) \arrow[ur] \arrow[r] & \mathrm{Spec}\, k \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r] \arrow[dr] & \mathrm{Spec}\, A \times_{\mathcal{X}} X \arrow[r] \arrow[d, "\text{smooth}"] & X \arrow[d, "x"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Spec}(A) \arrow[u, bend left, "id"] \arrow[u, phantom, "\circlearrowleft", shift left=3] \arrow[ur, bend left=60] \arrow[r] \arrow[d, equal] & \mathcal{X} \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}\, k
\end{tikzcd}$$

→ smooth

Examples of Artin stacks:

① **Quotient stacks**: S scheme

X noetherian S-scheme,

G affine smooth group S-scheme

$X \circlearrowleft G$, $[X/G]$ is an Artin stack.

Atlas for $[X/G]$:

$$\begin{tikzcd} & & X \arrow[r, "\pi"] & {[X/G]} \\ X \times G \arrow[r, "pr_1"] \arrow[d, "\text{action, } G\text{-equivariant}"'] & X \arrow[r, phantom, "\Longleftrightarrow"] & {[X/G](X)} \arrow[u, Leftrightarrow] & \\ X & & & \end{tikzcd}$$

smooth surjective

$U = S$-scheme

$$\begin{tikzcd} \mathcal{E} \cong X \times_{[X/G]} U \arrow[r] \arrow[d, "G\text{-bundle}"] & X \arrow[d, "\pi"] \\ U \arrow[r, "y"] & {[X/G]} \end{tikzcd}$$

universal family of G-bundle

smooth since G is smooth

y $\shortparallel$ a $\underline{G}$-bundle $\mathcal{E}$ over U + G-eq. map.

---

② $\mathcal{M}_g$, $\overline{\mathcal{M}_g}$ $g \geq 2$ are $\underline{DM}$ stacks.

③ $\mathrm{Bun}_n X$ = moduli stack of rank n vector bundles

X = smooth projective irreducible algebraic curve / Spec k

$$\mathrm{Bun}_n X = \coprod_{d \in \mathbb{Z}} \underbrace{\mathrm{Bun}_n^d X}_{\substack{\parallel \\ \text{connected} \\ \text{components} \\ \text{of } X}}$$

$\mathrm{Bun}_n^d X$ is a smooth, locally of finite type Artin stack.

- $$\mathrm{Bun}_n^d X = \coprod_{m\in\mathbb{Z}} \underbrace{R_m}_{\substack{\text{smooth}\\ \text{and locally}\\ \text{of finite}\\ \text{type}}}$$

$$P_{n,d}(x) := nx + d + n(1-g)$$

$$\forall m \geq 0, \quad \boxed{\mathrm{Quot}\left(\mathcal{O}_X^{P(m)}, P\right)} \underset{\substack{\text{open}\\ \text{subscheme}}}{\supset} R_m$$

- $\mathcal{F} \in$ Coherent sheaf on X

scheme $\leftarrow$ $\mathrm{Quot} : (Sch/S)^{op} \longrightarrow (Sets)$

parametrizing quotients of $\mathcal{F}$ with a fixed Hilbert polynomial P

$$R_m = \left\{ \mathcal{F} \in \mathrm{Quot}(\mathcal{O}_X^{P(m)}, P) \right.$$

$\mathcal{F}$ is a vector bundle,

$\left\{ \right.$ for every U-point of R_m defined by the family $\mathcal{O}_{X \times U}^{P(m)} \longrightarrow \mathcal{F} \to 0$

we have that

$$R^1(pr_2)_* \mathcal{F} = 0$$

and

$$(pr_2)_* : \mathcal{O}_{X \times U}^{P(m)} \xrightarrow{\cong} (pr_2)_* \mathcal{F}$$

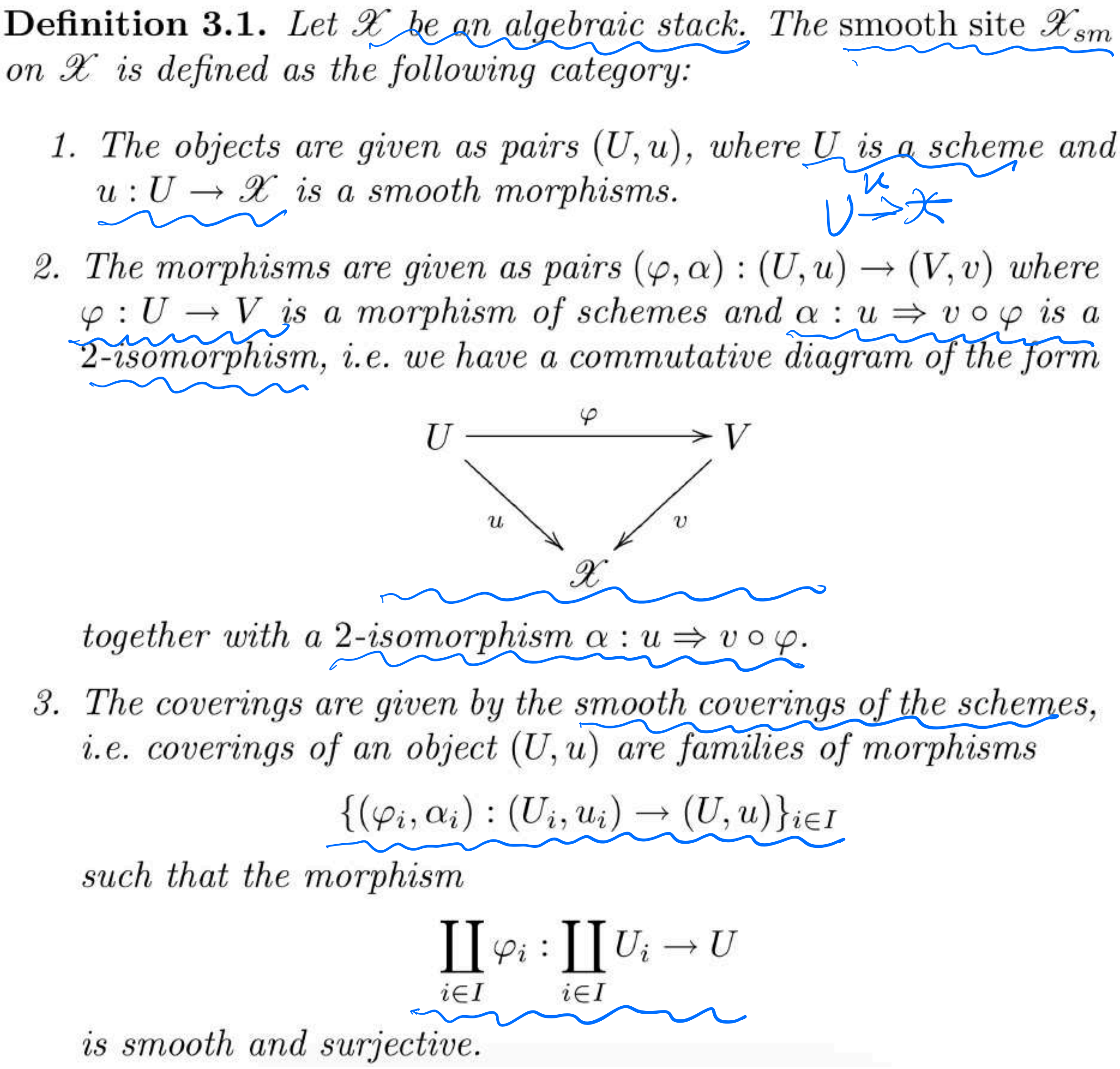
$$R_m \hookrightarrow \mathrm{Bun}_n^d X$$

Cohomology of algebraic stacks

GOAL. ℓ-adic cohomology of the moduli stack of vector bundles of fixed rank and degree on an algebraic curve

$$\begin{tikzcd}
\text{smooth site} & \text{sheaf} \arrow[l] \arrow[r, phantom, "\supset"] \arrow[d] & \text{vector bundle} \\
 & \text{cohomology} &
\end{tikzcd}$$

1. Smooth site



Definition 3.1. *Let $\mathscr{X}$ be an algebraic stack. The* smooth site $\mathscr{X}_{sm}$ *on $\mathscr{X}$ is defined as the following category:*

1. *The objects are given as pairs (U, u), where U is a scheme and $u : U \to \mathscr{X}$ is a smooth morphisms.*

   $U \xrightarrow{u} \mathscr{X}$

2. *The morphisms are given as pairs $(\varphi, \alpha) : (U, u) \to (V, v)$ where $\varphi : U \to V$ is a morphism of schemes and $\alpha : u \Rightarrow v \circ \varphi$ is a 2-isomorphism, i.e. we have a commutative diagram of the form*

$$\begin{tikzcd}
U \arrow[rr, "\varphi"] \arrow[dr, "u"'] & & V \arrow[dl, "v"] \\
& \mathscr{X} &
\end{tikzcd}$$

   together with a 2-isomorphism $\alpha : u \Rightarrow v \circ \varphi$.

3. *The coverings are given by the smooth coverings of the schemes, i.e. coverings of an object (U, u) are families of morphisms*

$$\{(\varphi_i, \alpha_i) : (U_i, u_i) \to (U, u)\}_{i \in I}$$

   such that the morphism

$$\coprod_{i \in I} \varphi_i : \coprod_{i \in I} U_i \to U$$

   is smooth and surjective.

Grothendieck topology on the smooth site as induced by the smooth topology on the category (Sch/S) of S-schemes

2. Sheaf

Definition 3.2. *Let $\mathscr{X}$ be an algebraic stack.* A sheaf $\mathcal{F}$ *on the smooth site $\mathscr{X}_{sm}$ is given by the following data:*

1. *For each object (U, u) of $\mathscr{X}_{sm}$, where U is a scheme and $u : U \to \mathscr{X}$ a smooth morphism, a sheaf $\mathcal{F}_{U,u}$ on U.*
2. *For each morphism $(\varphi, \alpha) : (U, u) \to (V, v)$ of $\mathscr{X}_{sm}$ a morphism of sheaves*

$$\theta_{\varphi,\alpha} : \varphi^* \mathcal{F}_{V,v} \to \mathcal{F}_{U,u}$$

satisfying the cocycle condition for composable morphisms, i.e. for each commutative diagram of the form

$\theta_{\psi,\beta} : \psi^* F_{W,w} \Rightarrow F_{V,v}$

$\varphi^*\theta_{\psi,\beta} : \varphi^*\psi^* F_{W,w} \Rightarrow \varphi^* F_{V,v}$

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "u"'] & V \arrow[r, "\psi"] \arrow[d, "v"] & W \arrow[dl, "w"] \\ & \mathscr{X} & \end{tikzcd}$$

together with 2-isomorphisms $\alpha : u \Rightarrow v \circ \varphi$ and $\beta : v \Rightarrow w \circ \psi$ we have that

$(\psi\circ\varphi)^* F_{W,w}$

$$\theta_{\varphi,\alpha} \circ \varphi^* \theta_{\psi,\beta} = \theta_{\psi\circ\varphi,\varphi_*\beta\circ\alpha}$$

We will focus on special sheaves

A sheaf $\mathcal{F}$ is called quasi-coherent (resp. coherent, resp. of finite type, resp. of finite presentation, resp. locally free) *if the sheaf $\mathcal{F}_{U,u}$ is quasi-coherent (resp. coherent, resp. of finite type, resp. of finite presentation, resp. locally free) for every morphism $u : U \to \mathscr{X}$, where U is a scheme.*

A sheaf $\mathcal{F}$ is called cartesian *if all morphisms $\theta_{\varphi,\alpha}$ are isomorphisms.*

* ① Consider affine schemes

② Glue

In general, it is not enough to consider just cartesian sheaves (not enough injectives)

BUT $\mathcal{C}_{\text{cartesian sheaves}}$ is a thick subcategory of $\mathcal{C}_{\text{sheaves}}$

i.e. full subcategory closed under kernels, quotients and extensions

3. Examples.

Definition 3.3. *A vector bundle on an algebraic stack $\mathscr{X}$ is a coherent sheaf $\mathcal{E}$ on $\mathscr{X}_{sm}$ such that all coherent sheaves $\mathcal{E}_{U,u}$ are locally free for every morphism $u : U \to \mathscr{X}$, where U is a scheme.*

Example 3.4. (Structure sheaf of an algebraic stack) Let $\mathscr{X}$ be an algebraic stack. The *structure sheaf* $\mathcal{O}_{\mathscr{X}}$ on $\mathscr{X}$ is defined by assembling the structure sheaves $\mathcal{O}_U$ of the schemes U for every smooth morphism $u : U \to \mathscr{X}$, i.e. by setting $(\mathcal{O}_{\mathscr{X}})_{U,u} = \mathcal{O}_U$. In this way we get a ringed site $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ on the algebraic stack $\mathscr{X}$ and we can define sheaves of $\mathcal{O}_{\mathscr{X}}$-modules, sheaves of quasi-coherent $\mathcal{O}_{\mathscr{X}}$-modules and, if $\mathscr{X}$ is locally noetherian, also sheaves of coherent $\mathcal{O}_{\mathscr{X}}$-modules [LMB00], Chap. 13 & 15.

Example 3.5. (Constant sheaf $\mathbb{Z}/n\mathbb{Z}$) Let $\mathscr{X}$ be an algebraic stack. Let $n \geq 1$ be a positive integer. The *constant sheaf* $(\mathbb{Z}/n\mathbb{Z})_{\mathscr{X}}$ is given by assembling the constant sheaves $(\mathbb{Z}/n\mathbb{Z})_{U,u} = (\mathbb{Z}/n\mathbb{Z})_U = \mathbb{Z}/n\mathbb{Z}$. It turns out that this is actually a cartesian sheaf on $\mathscr{X}$ [LMB00], 12.7.1 (ii).

$\mathcal{E}^{univ} \mapsto id_{\mathrm{Bun}^{n,d}_{X}}$

$Bun_X^{n,d}(Bun_X^{n,d}) \cong Hom_{(Schemes)}(Bun_X^{n,d}, Bun_X^{n,d})$

$Bun_X^{n,d}$ coarse moduli space

Example 3.7. (Universal vector bundle $\mathcal{E}^{univ}$ on $X \times \mathscr{B}un_X^{n,d}$) Let $\mathscr{B}un_X^{n,d}$ be the moduli stack of rank n and degree d vector bundles on a smooth projective irreducible algebraic curve X of genus $g \geq 2$. There exists a *universal vector bundle* $\mathcal{E}^{univ}$ on the algebraic stack $X \times \mathscr{B}un_X^{n,d}$, because via representability any morphism $U \to \mathscr{B}un_X^{n,d}$, where U is a scheme defines a family of vector bundles of rank n and degree d on the scheme X parametrized by U and the cocycle conditions can easily be checked for vector bundles. Similar, we get universal vector bundles for the moduli stacks $\mathscr{B}un_X^{ss,n,d}$ (resp. $\mathscr{B}un_X^{st,n,d}$) of semistable (resp. stable) vector bundles.

Example 3.8. (Equivariant sheaves) Let (Sch/S) be the category of S-schemes and X be a noetherian S-scheme. Let G be an affine smooth group S-scheme with a right action $\rho : X \times G \to X$ and consider the quotient stack $[X/G]$. Then any cartesian sheaf $\mathcal{F}$ on $[X/G]$ is the same as an G-equivariant sheaf on X.

4. Global sections + Sheaf cohomology

Let $\mathscr{X}$ be an algebraic stack and choose an atlas $u : U \to \mathscr{X}$ of $\mathscr{X}$. For cartesian sheaves $\mathcal{F}$ on $\mathscr{X}$ we define the global sections as the equalizer

$$\Gamma(\mathscr{X}, \mathcal{F}) := \mathrm{Ker}(\Gamma(U, \mathcal{F}) \rightrightarrows \Gamma(U \times_{\mathscr{X}} U, \mathcal{F})).$$

It is not hard to see that this definition does not depend on the choice of the atlas $u : U \to \mathscr{X}$ of $\mathscr{X}$ by first checking it on a covering and then on refinements.

$$\begin{tikzcd}
U \times_{\mathscr{X}} U \arrow[r] \arrow[d] & U \arrow[d, "u"] \\
U \arrow[r, "u"] & \mathscr{X}
\end{tikzcd}$$

Definition 3.10. *Let $\mathscr{X}$ be an algebraic stack and $\mathcal{F}$ a quasi-coherent sheaf on $\mathscr{X}_{sm}$. The set of* global sections *is defined as*

$$\Gamma(\mathscr{X},\mathcal{F}) := \{(s_{U,u}) : s_{U,u} \in H^0(U,\mathcal{F}_{U,u}), \theta_{\varphi,\alpha} s_{U,u} = s_{V,v}\}.$$

The functor

$$\Gamma(\mathscr{X},?) : \mathrm{Shv}(\mathscr{X}) \to (Sets)$$

is called the global section functor.

$$\begin{tikzcd} (U,u) \arrow[d] \\ (V,v) \end{tikzcd}$$

We can rephrase this by saying that the global sections are given as the limit

$$\Gamma(\mathscr{X},\mathcal{F}) = \lim_{\leftarrow} \Gamma(U,\mathcal{F}_{U,u})$$

$$\begin{tikzcd} \Gamma(\mathscr{X},\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(V,\mathcal{F}_{V,v}) \\ \Gamma(U,\mathcal{F}_{U,u}) \arrow[ur] & \end{tikzcd}$$

where the limit is taken over all atlases $u : U \to \mathscr{X}$ with transition functions given by the restriction maps $\theta_{\varphi,\alpha}$. Again, it is not hard to show that for cartesian sheaves the two notions of global sections coincide.

∀ Mod($\mathscr{X}$) of sheaves of $\mathcal{O}_{\mathscr{X}}$-modules and Ab($\mathscr{X}$) of sheaves of abelian groups on the site $\mathscr{X}_{sm}$ have enough injective objects.

Definition 3.11. *The* i-th smooth cohomology group *of the algebraic stack $\mathscr{X}$ with respect to a sheaf $\mathcal{F}$ of abelian groups on the smooth site $\mathscr{X}_{sm}$ is defined as*

$$H^i_{sm}(\mathscr{X},\mathcal{F}) := R^i\Gamma(\mathscr{X},\mathcal{F})$$

where the cohomology functor

$$H^i_{sm}(\mathscr{X},?) = R^i\Gamma(\mathscr{X},?) : \mathfrak{Ab}(\mathscr{X}) \to \mathfrak{Ab}$$

is the i-th right derived functor of the global section functor $\Gamma(\mathscr{X},?)$ with respect to $\mathscr{X}_{sm}$.

☆ Derived functor

$\mathcal{F} \to I^0 \to I^1 \to \cdots$ injective resolution

then $R^i\Gamma(\mathcal{F}) = H^i(\cdots \to \underset{0}{0} \to \Gamma(I^0) \to \Gamma(I^1) \to \cdots)$

5. simplicial interpretation

For cartesian sheaves we can give a simplicial interpretation of the sheaf cohomology of an algebraic stack $\mathscr{X}$ [LMB00], 12.4. Let $x : X \to \mathscr{X}$ be an atlas. As the diagonal morphism of an algebraic stack is representable, we obtain by taking iterated fiber products of the atlas with itself

$$\begin{tikzcd} X \times_{\mathscr{X}} X \arrow[r] \arrow[d] & X \arrow[d, "x"] \\ X \arrow[r, "x"] & \mathscr{X} \end{tikzcd}$$

a simplicial scheme $X_\bullet = \{X_i\}_{i \geq 0}$ over $\mathscr{X}$ with layers

$$X_i = X \times_{\mathscr{X}} X \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} X$$

given by the $(i+1)$-fold iterated fiber product of the atlas with itself.

A simplicial scheme $X_\bullet$ over $\mathscr{X}$ can simply be interpreted as a functor

$$X_\bullet : \Delta^{op} \to (Sch/\mathscr{X})$$

$[i] \mapsto X_i$

where Δ^{op} is the category with objects finite sets $[n] = \{0, 1, \ldots n\}$ and morphisms order preserving maps and $(Sch/\mathscr{X})$ is the category of schemes over the algebraic stack $\mathscr{X}$, i.e. the category of schemes X together with morphisms $x : X \to \mathscr{X}$.

Now let $\mathcal{F}$ be a sheaf on $\mathscr{X}$. This defines a sheaf $\mathcal{F}_\bullet$ on the simplicial scheme $X_\bullet$, i.e. a sheaf $\mathcal{F}_i$ on all schemes X_i together with morphisms for all simplicial maps $f : [m] \to [n]$ of the form f^* :

$X_\bullet(f)^*\mathcal{F}_n \to \mathcal{F}_m$. We call a sheaf on a simplicial scheme *cartesian* if all morphisms f^* are isomorphisms. If we start with a cartesian sheaf $\mathcal{F}$ on $\mathscr{X}$, we get a cartesian sheaf $\mathcal{F}_\bullet$ on the simplicial scheme $X_\bullet$. In this way we get a functor $\mathrm{Shv}(\mathscr{X}) \to \mathrm{Shv}(X_\bullet)$.

Conversely, for any smooth morphism $u : U \to \mathscr{X}$ a sheaf $\mathcal{F}_\bullet$ on the simplicial scheme $X_\bullet$ gives a sheaf on the covering $U \times_{\mathscr{X}} X \to U$ via taking global sections and by assembling them to a sheaf on $\mathscr{X}$. Again starting with a cartesian sheaf $\mathcal{F}_\bullet$ on $X_\bullet$ gives a cartesian sheaf on $\mathscr{X}$.

We can define cohomology of sheaves of abelian groups on simplicial schemes generalizing the classical homological approach for sheaf cohomology on schemes [Fri82].

6. Spectral Sequence

Theorem 3.12. *Let $\mathscr{X}$ be an algebraic stack and $\mathcal{F}$ be a cartesian sheaf of abelian groups on $\mathscr{X}$. Let $x : X \to \mathscr{X}$ be an atlas and $\mathcal{F}_\bullet$ the induced sheaf on the simplicial scheme $X_\bullet$ over $\mathscr{X}$. Then there is a convergent spectral sequence*

$$E_1^{p,q} \cong H^p_{sm}(X_q, \mathcal{F}_q) \Rightarrow H^{p+q}_{sm}(\mathscr{X}, \mathcal{F}).$$

which is functorial with respect to morphisms $F : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks.

$$\begin{tikzcd} \overline{\mathscr{X}} \arrow[r] \arrow[d] & \mathrm{Spec}(\overline{\mathbb{F}}_q) \arrow[d] \\ \mathscr{X} \arrow[r] & \mathrm{Spec}(\mathbb{F}_q) \end{tikzcd}$$

7. l-adic cohomology

Example 3.13. (l-adic smooth cohomology) Let $\mathscr{X}$ be an algebraic stack defined over the field $\mathbb{F}_q$ of characteristic p. Via base change we get an associated algebraic stack $\overline{\mathscr{X}}$ over the algebraic closure $\overline{\mathbb{F}}_q$ by setting

$$\overline{\mathscr{X}} = \mathscr{X} \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q).$$

Let l be a prime number different from p. The *l-adic smooth cohomology* of the algebraic stack $\overline{\mathscr{X}}$ is defined as

$$H^*_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) = \varprojlim H^*_{sm}(\overline{\mathscr{X}}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

8. Properties

$\mathbb{Z}_l$

Theorem 3.14. *We have the following properties:*

(1.) *(Künneth decomposition) Let $\mathscr{X}$ and $\mathscr{Y}$ be algebraic stacks. There is a natural isomorphism of graded $\mathbb{Q}_l$-algebras*

$$H^*_{sm}(\overline{\mathscr{X}} \times \overline{\mathscr{Y}}, \mathbb{Q}_l) \cong H^*_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) \otimes H^*_{sm}(\overline{\mathscr{Y}}, \mathbb{Q}_l).$$

(2.) *(Gysin sequence) Let $\mathscr{Z} \hookrightarrow \mathscr{X}$ be a closed embedding of algebraic stacks of codimension c. There is a long exact sequence*

$$\cdots \to H^{i-2c}_{sm}(\overline{\mathscr{Z}}, \mathbb{Q}_l(c)) \to H^i_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) \to H^i_{sm}(\overline{\mathscr{X} \backslash \mathscr{Z}}, \mathbb{Q}_l) \to \cdots$$

In particular, $H^i_{sm}(\overline{\mathscr{X}}, \mathbb{Q}_l) \xrightarrow{\cong} H^i_{sm}(\overline{\mathscr{X} \backslash \mathscr{Z}}, \mathbb{Q}_l)$ is an isomorphism in the range $i < 2c - 1$.

9. Example

Example 3.15. (Cohomology of the classifying stack $\mathscr{B}\mathbb{G}_m$) Let $\mathbb{G}_m$ be the multiplicative group over $\mathrm{Spec}(\mathbb{F}_q)$. The quotient morphism $\mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ is a principal $\mathbb{G}_m$-bundle and we have a cartesian diagram of the form

$$\begin{tikzcd}
\mathbb{A}^n - \{0\} \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{F}_q) \arrow[d] \\
\mathbb{P}^{n-1} \arrow[r, "\pi"] & \mathscr{B}\mathbb{G}_m
\end{tikzcd}$$

The fiber of the morphism π is $\mathbb{A}^n - \{0\}$ and we can employ the Leray spectral sequence

$$E_2^{p,q} \cong \boxed{H^p_{sm}(\overline{\mathbb{P}}^{n-1}, R^q\pi_*\mathbb{Q}_l)} \Rightarrow \boxed{H^*_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l)}$$

and because $R^0\pi_*\mathbb{Q}_l \cong \mathbb{Q}_l$ and $R^q\pi_*\mathbb{Q}_l = 0$ if $q \leq 2n-1$ it follows for $q \leq 2n-1$ that

$$H^q_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong H^q_{sm}(\overline{\mathbb{P}}^{n-1}, \mathbb{Q}_l)$$

and therefore

$$H^*_{sm}(\overline{\mathscr{B}\mathbb{G}_m}, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1]$$

where c_1 is a generator of degree 2 given as the Chern class of the universal bundle $\mathcal{E}^{univ}$ on the classifying stack $\mathscr{B}\mathbb{G}_m$.

10. Gerbe

Definition 3.16. *A morphism $F : \mathscr{X} \to \mathscr{Y}$ of algebraic stacks is a* gerbe *over $\mathscr{Y}$ if the following holds:*

1. *F is locally surjective, i.e. for any morphism $U \to \mathscr{Y}$ from a scheme, there exists a covering $U' \to U$ such that the morphism $U' \to \mathscr{Y}$ can be lifted to a morphism $U' \to \mathscr{X}$, i.e.*

$$\begin{tikzcd} & \mathscr{X} \arrow[d, "F"] \\ U' \arrow[ur] \arrow[r] & \mathscr{Y} \end{tikzcd}$$

$$\begin{tikzcd} & & \mathscr{X} \arrow[d] \\ U' \arrow[urr, dashed] \arrow[r] & U \arrow[r] & \mathscr{Y} \end{tikzcd}$$

2. *All objects in a fiber of F are locally isomorphic, i.e. if $u_1, u_2 : U \to \mathscr{X}$ are objects of $\mathscr{X}(U)$ such that $F(u_1) \cong F(u_2)$, then there exists a covering $U' \to U$ such that $u_1|_{U'} \cong u_2|_{U'}$.*

A gerbe $F : \mathscr{X} \to \mathscr{Y}$ is a $\mathbb{G}_m$-gerbe *if for all morphisms $u : U \to \mathscr{X}$ the relative automorphism group* $\mathrm{Aut}_{\mathscr{Y}}(u)$ *is canonically isomorphic to $\mathbb{G}_m(U)$.*

We can think of a $\mathbb{G}_m$-gerbe over a scheme Y as a $\mathscr{B}\mathbb{G}_m$-bundle over Y, i.e. a bundle over Y with fiber $\mathscr{B}\mathbb{G}_m$.

11. Examples

Example 3.17. As mentioned before, there is a morphism of stacks

$$F : \mathscr{B}un_X^{st,n} \to \mathrm{Bun}^{st,n}$$

where $\mathscr{B}un_X^{st,n}$ is the moduli stack of stable vector bundles of rank n on an algebraic curve X with coarse moduli space $\mathrm{Bun}_X^{st,n}$. The morphism F has the following property: For any morphism $U \to \mathrm{Bun}_X^{st,n}$ of schemes there exists an étale covering $U' \to U$ such that the morphism $U' \to \mathrm{Bun}_X^{st,n}$ lifts to a morphism $U' \to R^{st,n}$ and so it lifts to the moduli stack $\mathscr{B}un_X^{st,n} = [R^{st,n}/GL_N]$.

Therefore F is a gerbe and because all automorphisms of stable bundles are given by scalars the fiber of F is isomorphic to $\mathscr{B}\mathbb{G}_m$, i.e. F is actually a $\mathbb{G}_m$-gerbe.

In general, a morphism of quotient stacks of the form

$$F : [R/GL_N] \to [R/PGL_N]$$

is a $\mathbb{G}_m$-gerbe. This is useful in order to compare "stacky" quotients with GIT quotients.

quotient stack GIT quotient

12. Triviality

Proposition 3.18. *Let $F : \mathscr{X} \to \mathscr{Y}$ be a $\mathbb{G}_m$-gerbe. Then the following are equivalent:*

1. *The $\mathbb{G}_m$-gerbe F is trivial, i.e. we have a splitting of algebraic stacks*
$$\mathscr{X} \cong \mathscr{Y} \times \mathscr{B}\mathbb{G}_m.$$
2. *The morphism F has a section.*

13. Example

Example 3.19. There is also a morphism of stacks

$$F : \mathscr{B}un_X^{st,n,d} \to \mathrm{Bun}_X^{st,n,d}$$

where $\mathscr{B}un_X^{st,n,d}$ is the moduli stack of stable vector bundles of rank n and degree d on X and $\mathrm{Bun}_X^{st,n,d}$ its coarse moduli space, given again as a scheme via GIT methods. A section of the morphism F is a vector bundle on $X \times \mathrm{Bun}_X^{st,n,d}$ such that the fiber over every geometric point of $\mathrm{Bun}_X^{st,n,d}$ lies in the isomorphism class of stable bundles defined by this geometric point. Such a vector bundle is also called a *Poincaré family*.

14. l-adic cohomology of moduli stack

In this section we will determine the l-adic cohomology algebra of the moduli stack $\mathscr{B}un_X^{n,d}$ of vector bundles of rang n and degree d on a smooth projective irreducible algebraic curve X over the field $\mathbb{F}_q$.

Let us recall the l-adic cohomology algebra of the moduli stack

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l) = \varprojlim H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Z}/l^m\mathbb{Z}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

l-adic cohomology of { ① BGLn of all rank n v.b.
② algebraic curve X

②

Theorem 4.1 (Weil, Deligne)**.** *Let X be a smooth projective curve of genus g over $\mathbb{F}_q$ and $\overline{X} = X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ the associated curve over the algebraic closure $\overline{\mathbb{F}}_q$. Then we have*

$$H^0_{et}(\overline{X};\mathbb{Q}_l) = \mathbb{Q}_l \cdot 1$$

$$H^1_{et}(\overline{X};\mathbb{Q}_l) = \bigoplus_{i=1}^{2g} \mathbb{Q}_l \cdot \alpha_i$$

$$H^2_{et}(\overline{X};\mathbb{Q}_l) = \mathbb{Q}_l \cdot [\overline{X}]$$

$$H^i_{et}(\overline{X};\mathbb{Q}_l) = 0, \ if\ i \geq 3$$

where $[\overline{X}]$ is the fundamental class and the α_i are eigenclasses under the action of the geometric Frobenius morphism

$$\overline{F}^*_X : H^*_{et}(\overline{X},\mathbb{Q}_l) \to H^*_{et}(\overline{X},\mathbb{Q}_l)$$

given as

$$\overline{F}^*_X(1) = 1$$

$$\overline{F}^*_X([\overline{X}]) = q[\overline{X}]$$

$$\overline{F}^*_X(\alpha_i) = \lambda_i\alpha_i \ (i = 1, 2, \ldots 2g)$$

where $\lambda_i \in \overline{\mathbb{Q}}_l$ is an algebraic integer with $|\lambda_i| = q^{1/2}$ for any embedding of λ_i in $\mathbb{C}$.

①

The other ingredient in the determination of the l-adic cohomology algebra of $\mathscr{B}un_X^{n,d}$ will be the l-adic cohomology of the classifying stack $\mathscr{B}GL_n$ of all rank n vector bundles. Let $\overline{\mathscr{B}GL}_n = \mathscr{B}GL_n \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{\mathbb{F}}_q)$ be the associated classifying stack over the algebraic closure $\overline{\mathbb{F}}_q$. We also have a geometric Frobenius morphism

$$\overline{F}^*_{\mathscr{B}GL_n} : H^*_{sm}(\overline{\mathscr{B}GL}_n, \mathbb{Q}_l) \to H^*_{sm}(\overline{\mathscr{B}GL}_n, \mathbb{Q}_l).$$

The l-adic cohomology algebra of $\mathscr{B}GL_n$ and the action of the Frobenius morphism $\overline{F}_{\mathscr{B}GL_n}$ is completely determined by the following theorem [Beh93].

Theorem 4.2. *There is an isomorphism of graded $\mathbb{Q}_l$-algebras*

$$H^*_{sm}(\overline{\mathscr{B}GL}_n, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \dots c_n]$$

and the geometric Frobenius morphism $\overline{F}^*_{\mathscr{B}GL_n}$ acts as follows

$$\overline{F}^*_{\mathscr{B}GL_n}(c_i) = q^i c_i \ \ (i \geq 1).$$

where the c_i are the Chern classes of the universal vector bundle $\tilde{\mathcal{E}}^{univ}$ of rank n over the classifying stack $\mathscr{B}GL_n$.

15. How to combine ① and ②

We will embark now on the determination of the l-adic cohomology algebra of the moduli stack $\mathscr{B}un_X^{n,d}$. Let $\mathcal{E}^{univ}$ be the universal vector bundle of rank n and degree d over the algebraic stack $\overline{X} \times \overline{\mathscr{B}un}_X^{n,d}$. Via representability it gives a morphism of stacks

$$u : \overline{X} \times \overline{\mathscr{B}un}_X^{n,d} \to \overline{\mathscr{B}GL}_n.$$

The universal vector bundle $\mathcal{E}^{univ}$ has Chern classes given as

$$c_i(\mathcal{E}^{univ}) = u^*(c_i) \in H^{2i}_{sm}(\overline{X} \times \overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l).$$

Fixing a basis $1 \in H^0_{sm}(\overline{X}, \mathbb{Q}_l)$, $\alpha_j \in H^1_{sm}(\overline{X}, \mathbb{Q}_l)$ with $j = 1, \dots 2g$ and $[\overline{X}] \in H^2_{sm}(\overline{X}, \mathbb{Q}_l)$ we get the following Künneth decomposition of Chern classes:

$$c_i(\mathcal{E}^{univ}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [\overline{X}] \otimes b_{i-1}.$$

where the classes $c_i \in H^{2i}_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l)$, $a_i^{(j)} \in H^{2i-1}_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l)$ and $b_{i-1} \in H^{2(i-1)}_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l)$ are the so-called *Atiyah-Bott classes.*

16. l-adic cohomology of moduli stack

Theorem 4.3. *Let X be a smooth projective irreducible algebraic curve of genus $g \geq 2$ over the field $\mathbb{F}_q$ and $\mathscr{B}un_X^{n,d}$ be the moduli stack of vector bundles of rank n and degree d on X. There is an isomorphism of graded $\mathbb{Q}_l$-algebras*

$$H^*_{sm}(\overline{\mathscr{B}un}_X^{n,d}, \mathbb{Q}_l) \cong \mathbb{Q}_l[c_1, \dots, c_n] \otimes \mathbb{Q}_l[b_1, \dots, b_{n-1}]$$
$$\otimes \Lambda_{\mathbb{Q}_l}(a_1^{(1)}, \dots, a_1^{(2g)}, \dots, a_n^{(1)}, \dots, a_n^{(2g)}).$$

① right ≤ left (induction)

② same

Classical Weil Conjectures

$\mathbb{Q}$

$\mathbb{Z}$

$X \subseteq \mathbb{CP}^n$ m-dim'l smooth complex proj. variety defined over a number ring R.

$\mathfrak{m} \subset R$ maximal ideal, then

$\boxed{R/\mathfrak{m} = \mathbb{F}_q}$ [e.g.: $R = \mathbb{Z}$, $\mathfrak{m} = p\mathbb{Z}$, $R/\mathfrak{m} = \mathbb{F}_p$].

$\mathbb{Z}[\sqrt{2}] \subseteq \mathbb{Q}(\sqrt{2})$

$\downarrow$ number ring

Now reduce eqns. of X mod $\mathfrak{m}$ to get a proj. variety $X_m / \mathbb{F}_q$

$x^2 - 2x + 3$

$$\boxed{X_m} \subset \mathbb{P}^n_{\mathbb{F}_q}.$$

$p = 5$

$p = 2$: $x^2 + 1$

$\boxed{\overline{X_m}}$ = associated proj. variety over $\overline{\mathbb{F}_q}$ defined by the same eqns. as for X_m but viewed over $\overline{\mathbb{F}_q}$.

$\boxed{N_r} = \boxed{\# X(\boxed{\mathbb{F}_{q^r}})} = \#$ of points in $\underline{\overline{X}_m}$ of the form $(\underline{x_0 : \cdots : x_n}$ such that $x_j \in \mathbb{F}_{q^r}$.

Defn: Let $\underline{X}$ be an m-dim'l smooth complex proj. variety defined over a number ring $\underline{R}$. The zeta function of the associated variety $\underline{\overline{X}_m}$ is defined as

$$\underline{Z(t)} = Z(\overline{X}_m, t) = \boxed{\exp\left(\sum_{r \geq 1} N_r \frac{t^r}{r}\right)}$$

$$= 1 + \left(\sum_{r \geq 1} N_r \frac{t^r}{r}\right) + \frac{1}{2!}\left(\sum_{r \geq 1} N_r \frac{t^r}{r}\right)^2$$

$$+ \cdots \in \mathbb{Q}[|t|]$$

Example: $X = \mathbb{CP}^m$ is defined over $R = \mathbb{Z}$. Let $\mathfrak{m} = p\mathbb{Z}$, $R/\mathfrak{m} = \mathbb{F}_p$.

$$\overline{\mathbb{P}^m_{\mathfrak{m}}}(\mathbb{F}_{p^r}) = \frac{(p^r)^{m+1} - 1}{p^r - 1}$$

$\mathbb{P}^m(\mathbb{F}_{p^r})$ variety over $\mathbb{F}_p$

$$\boxed{N_r = 1 + p^r + p^{2r} + \cdots + p^{mr}}$$

$$\Rightarrow Z(t) = \exp\left(\sum_{r \geq 1} (1 + p^r + \cdots + p^{mr}) \frac{t^r}{r}\right)$$

$$\boxed{= \frac{1}{(1-t)(1-pt)(1-p^2t)\cdots(1-p^m t)}}$$

Thm (Weil conjecturas) Let $Z(t) = Z(X,t)$ be the zeta fn. of an m-dim'l smooth complex proj. variety X over an algebraic number ring R. Then

- $$Z(t) = \frac{P_1(t)\,P_3(t)\cdots P_{2m-1}(t)}{P_0(t)\,P_2(t)\cdots P_{2m}(t)}$$

Rationality

$$= \prod_j \left(\prod_{1\le i\le \dim H_j(X,\mathbb{C})} (1-\alpha_{ji}t) \right)^{(-1)^{j+1}}$$

with $P_0(t) = 1-t$, $P_{2m}(t) = 1-q^m t$

and for $1 \le j \le 2m-1$,

$$\boxed{P_j(t) = \prod_{1\le j\le \dim H_j(X,\mathbb{C})} (1-\alpha_{ji}t)}$$

where the α_{ji} are algebraic integers w/ $\boxed{|\alpha_{ji}| = q^{-j/2}}$, i.e., the zeta function $Z(t)$ determines uniquely the polynomials $P_j(t)$ and hence the Betti numbers $\boxed{\dim H_j(X, \mathbb{C}) = \deg P_j(t)}$

Let $\boxed{\chi = \chi(X) = \sum_j (-1)^j \dim H_j(X, \mathbb{C})}$

Euler characteristic

Then we have the functional eqn-

Functional equation

$$\boxed{Z\left(\frac{1}{q^m t}\right) = \pm\, q^{\chi/2} t^{\chi} Z(t).}$$

hardest one

Weil conjectures as an analogue of Riemann Hypothesis for algebraic curves

---

Let X be as before. Let $\mathfrak{p}$ denote a prime divisor of X, i.e., an equivalence class of points of $\overline{X}_m$ modulo conjugation over $\underline{\mathbb{F}_q}$. Define norm as:

$$\underline{\text{Norm}(\mathfrak{p})} = \underline{q^{\deg(\mathfrak{p})}}$$

$\deg(\mathfrak{p})$ = # points in the equivalence of $\mathfrak{p}$.

Now $\mathbb{F}_{q^i} \subset \mathbb{F}_{q^j} \iff i \mid j$ gives:

$$N_r = \#X(\mathbb{F}_{q^r}) = \sum_{\substack{\mathfrak{p}: \\ \deg(\mathfrak{p}) \mid r}} \deg(\mathfrak{p})$$

$x^2+1 \to +i, -i$ $\boxed{\{i,-i\}}$

Let $t = q^{-s}$,

$$\underline{Z(q^{-s})} = \exp\left(\sum_{r\geq 1} N_r \frac{q^{-rs}}{r}\right)$$

$$= \exp\left(\sum_{r\geq 1} \sum_{\substack{\mathfrak{p}: \\ \deg(\mathfrak{p})|r}} \frac{\deg(\mathfrak{p})\, \text{Norm}(\mathfrak{p})^{-rs/\deg\mathfrak{p}}}{r}\right)$$

$$= \exp\left(\sum_{\mathfrak{p}} \underbrace{\sum_{i} \frac{\text{Norm}(\mathfrak{p})^{-si}}{i}}_{-\text{Log}(1-\text{Norm}(\mathfrak{p})^{-s}}\right)$$

$$= \underline{\prod_{\mathfrak{p}} \frac{1}{1-\text{Norm}(\mathfrak{p})^{-s}}}$$

$$\frac{1}{1-p^{-s}} = \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}\cdots\right)$$

Looks very much like the classical Riemann zeta fn

$$\boxed{\underline{\zeta(s)} := \sum_{n\geq 1}\frac{1}{n^s} \overset{\text{Euler}}{=} \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}}$$

Riemann Hypothesis:

If $\zeta(s)=0$, then $\mathrm{Re}(s)=\frac{1}{2}$.

Let X be a complex smooth proj. alg. curve. Then $1-\alpha_{ji}t=0$

$\frac{1}{\alpha_{ji}}=t$

• $$Z(t)=\frac{\prod_{1\leq i\leq \dim H_1(X,\mathbb{C})}(1-\underline{\alpha_{ji}t})}{(1-t)(1-qt)}$$

So $\overbrace{|\alpha_{ji}|=q^{-1/2}} \implies$ If $Z(t)=0$, then $|t|=q^{1/2}$, i.e., if $\underset{\overset{\|}{0}}{Z(q^{-s})}$ then $\mathrm{Re}(s)=\frac{1}{2}$.

Thus, Weil conjectures for alg. curves is an analogue of RH.

Lefschetz Trace Formula:

Defn: Let X be as before. The Frobenius morphism f is defined as:

$$f : \overline{X}_m \longrightarrow \overline{X}_m$$

$$(x_0 : \cdots : x_n) \mapsto (x_0^q : \cdots : x_n^q)$$

$L(f^r) := \#$ fixed points of f^r.

Then easily

$$N_r = \# X(\mathbb{F}_{q^r}) = L(f^r)$$

For $x \in \overline{\mathbb{F}_q}$, $x^q = x \Leftrightarrow x \in \mathbb{F}_q$

Thm (Grothendieck - Deligne):

$$\underline{L(f^r)} = \sum_{1 \leq j \leq 2m} (-1)^j \operatorname{Tr}(\; \cdot \;)$$

$$(f^r)^* : H^j_{et}(\overline{X}_m, \mathbb{Q}_\ell) \longrightarrow H^j_{et}(\overline{X}_m, \mathbb{Q}_\ell)$$

where ℓ is a prime $\neq p$.

The cohomology in the trace formula is the ℓ-adic étale cohomology of $\overline{X}_m$:

$$H^*_{et}(\overline{X}_m, \mathbb{Q}_\ell) := \left(\varprojlim H^*_{et}(\overline{X}_m, \mathbb{Z}/\ell^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell .$$

$$Z(t) = \exp\left(\sum_{r\geq 1} \frac{N_r t^r}{r}\right)$$

$$= \exp\left(\sum_{r\geq 1} \frac{L(f^r)\, t^r}{r}\right) \to$$

$$= \exp\left(\sum_{r\geq 1}\left(\sum_{0\leq j\leq 2m} (-1)^j \mathrm{Tr}(f^*)\right)\frac{t^r}{r}\right)$$

$$= \prod_{j=0}^{2m} \det\left(1 - t f^* : H^j \to H^j\right)^{(-1)^{j+1}}$$

$$= \frac{P_1(t)\, P_3(t) \cdots P_{2m-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2m}(t)}$$

$\sum_j (-1)^{j+1}\mathrm{Tr}(f^*) \leftarrow \sum_{r\geq 1}\frac{t^r}{r}$ $\log(1-t)$

$\exp(\mathrm{Tr}(f^*)\log(1-t))$

$\boxed{\alpha_{ji}}$ = eigenvalues of induced Frobenius morphism in l-adic cohomology

$$P_j(t) = \det(1-tf^*) = \prod_{1\leq i\leq \dim H^j_{et}} (1-\alpha_{ji}t)$$

Frobenius morphisms for the moduli stack on $\overline{\mathrm{Bun}}^{X}_{r,d}$

X = smooth proj. alg. curve of genus g / $\mathbb{F}_q$

- **Geometric Frobenius of X:**

$$F_X : (X, \mathcal{O}_X) \longrightarrow (X, \mathcal{O}_X)$$

$$(\mathrm{id}_X, f \mapsto f^q)$$

$$\rightsquigarrow F_{\overline{X}} : \overline{X} \longrightarrow \overline{X}$$

$$\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q \ , \ F_{\overline{X}} = F_X \times \mathrm{id}_{\mathrm{Spec}(\overline{\mathbb{F}}_q)}$$

$\forall\ \boxed{U \in Sch/\mathbb{F}_q}$,

$$\overline{Bun}_X^{n,d}(U) \longrightarrow \overline{Bun}_X^{n,d}(U)$$

$\textcircled{\mathcal{E}}$

$\bar{X}\times U \xrightarrow{\bar{F}\times id} \bar{X}\times U$

v.b

$$\mathcal{E} \longmapsto \underline{\underline{\bar{F}^*(\mathcal{E}) = (\bar{F}_X \times id_U)^*(\mathcal{E})}}$$

$$\rightsquigarrow \varphi : \overline{Bun}_X^{n,d} \longrightarrow \overline{Bun}_X^{n,d}$$

induced geometric Frobenius morphism

Genuine geometric Frobenius morphism:

(raise sections to the q-th power using atlas): $F_{Bun_X^{n,d}} : Bun_X^{n,d} \to Bun_X^{n,d}$

$$\rightsquigarrow \boxed{\overline{F}_{\mathrm{Bun}_X^{n,d}}} = F_{\mathrm{Bun}_X^{n,d}} \times \mathrm{id}_{\mathrm{Spec}(\overline{\mathbb{F}_q})}$$

$$\overline{\mathrm{Bun}}_X^{n,d} = \mathrm{Bun}_X^{n,d} \times \overline{\mathbb{F}_q}$$

Arithmetic Frobenius:

- $\mathrm{Frob} : \overline{\mathbb{F}_q} \longrightarrow \overline{\mathbb{F}_q}$

  $a \longmapsto a^q.$

- $\mathrm{Frob}_{\mathrm{Spec}(\overline{\mathbb{F}_q})} : \mathrm{Spec}(\overline{\mathbb{F}_q}) \longrightarrow \mathrm{Spec}(\overline{\mathbb{F}_q})$

$$\rightsquigarrow \underline{\Psi :=} \ \mathrm{id}_{\mathrm{Bun}_X^{n,d}} \times \mathrm{Frob}_{\mathrm{Spec}(\overline{\mathbb{F}_q})}$$

is an endomorphism of $\overline{\mathrm{Bun}}_X^{n,d}$.

Recall: $H^*(\overrightarrow{\mathrm{Bun}}_{n,d}, \mathbb{Q}_\ell)$

$$\begin{gathered} \| \\ \underbrace{\mathbb{Q}_\ell[c_1, \dots, c_n] \otimes \mathbb{Q}_\ell[b_1, \dots, b_{n-1}] \otimes \Lambda_{\mathbb{Q}_\ell}(a_1^{(1)}, \dots, a_1^{(2g)}, \dots, a_n^{(1)}, \dots, a_n^{(2g)})}_{\text{Atiyah Bott classes}} \end{gathered}$$

Induced geometric	Geometric	Arithmetic
$c_i \mapsto c_i$	$c_i \mapsto q^i c_i$	$\boxed{c_i \mapsto q^{-i} c_i}$
$b_i \mapsto q b_i$	$b_i \mapsto q^{i-1} b_i$	$b_i \mapsto q^{-i+1} b_i$
$a_i^{(j)} \mapsto \lambda_j a_i^{(j)}$	$a_i^{(j)} \mapsto \lambda_j^{-1} a_i^{(j)}$	$a_i^{(j)} \mapsto \lambda_j a_i^{(j)}$

Geometric frobenius is inverse of arithmetic Frobenius

$\mathcal{X}$ = alg. stack

$$\begin{tikzcd}
X \arrow[d, "\text{atlas}", "\pi \leftarrow"'] \\
\mathcal{X}
\end{tikzcd}$$

$$\operatorname{rel\,dim}(X/\mathcal{X}) = \dim(X) - \dim(\mathcal{X})$$

$$\dim(\mathcal{X}) := \dim X - \operatorname{rel.dim}(X/\mathcal{X})$$

<u>eg</u>: $\mathcal{X} = [X/G]$

$$\begin{tikzcd}
X \times G \arrow[r, phantom, "="] & \mathcal{E} \arrow[r] \arrow[d, "\text{rel.dim}"] & X \arrow[d] \\
& X \arrow[r] & X/G
\end{tikzcd}$$

$$\dim([X/G]) = \dim X - \dim(G)$$

Eg: $B\mathbb{G}_m = [*/\mathbb{G}_m]$

$$\dim(B\mathbb{G}_m) = \dim(*) - \dim(\mathbb{G}_m$$

$$= -1$$

$$\# B\mathbb{G}_m(\operatorname{Spec}\mathbb{F}_q) = \sum_{x \in \underbrace{[B\mathbb{G}_m(\operatorname{Spec}\mathbb{F}_q)]}} \frac{1}{|\mathrm{Aut}(x)|}$$

$B\mathbb{G}_m(\operatorname{Spec}\mathbb{F}_q) = \mathbb{G}_m$- bundles over $\operatorname{Spec}\mathbb{F}_q$

$\frac{1}{|\mathrm{Aut}(\mathbb{G}_m \times \operatorname{Spec}\mathbb{F}_q)|}$ = line bundles over $\operatorname{Spec}\mathbb{F}_q$

$$= \frac{1}{|\mathbb{G}_m|} = \frac{1}{q-1}$$

$$q^{-1} \sum_{j} (-1)^{j} \mathrm{Tr}(f^{*}) = \frac{1}{q-1}$$

$$\sum_{j} (-1)^{j} \mathrm{Tr}(f^{*}) = \frac{q}{q-1}$$

Last Time:

$$H^{*}(\overline{BGL_n}, \mathbb{Q}_\ell)$$

$$\| $$

$$\mathbb{Q}_\ell[c_1, \ldots, c_n]$$

$$H^{*}(\overline{B\mathbb{G}_m}, \mathbb{Q}_\ell) = \overbrace{\mathbb{Q}_\ell[c_1]}^{\text{polynomial ring}}$$

$$c_1 \mapsto q^{-1} c_1$$

$$c_1^2 \mapsto q^{-2} c_1^2$$

$$\vdots$$

$$\sum_{i=0}^{\infty} \frac{1}{q^i}$$

$$P_{\mathrm{Bun}_X^{n,d}}(t) = \sum_j \left(\dim H^j(\overline{\mathrm{Bun}}_{X}^{n,d}, \mathbb{Q}_\ell)\right) t^j$$