Example of geometric Langlands

Galois Objects $\xleftrightarrow[\text{original}]{\text{local}}$ Automorphic

$Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ Modular form

$\downarrow$ study automorphism

$GL(n, \mathbb{C})$

$\mathbb{Q} = Frac(\mathbb{Z})$

$\Big\downarrow$ replace by

$k[x]$ or $k(x)$

$\shortparallel$

$k(\mathbb{P}^1_k)$

Now

Galois $\xleftrightarrow[\text{global}]{\text{geometric}}$ Automorphic

$\pi_1(X) \Rightarrow GL_n(n, \mathbb{C})$ $\mathbb{C}(Bun_n(X))$

$k = \mathbb{F}_q$ functions on iso-classes of v.b.

$k = \mathbb{C}$

$X/\mathbb{C}$ geometric Langlands/$\mathbb{C}$ is a relation between

$\underbrace{Conn_r}_{\text{Ex. dim}=2}(X) \longleftrightarrow \underbrace{Bun_r}_{\text{e.g. dim}=1}(X)$

$\shortparallel$

$\{(E, D: E \to E \otimes \Omega^1_X)\}$

× no isomorphism

✓ turn to equivalence of categories

$QCoh(\underset{\text{Nilcone}}{Conn_n(X)}) \simeq \mathcal{D}\text{-mod}(Bun_n(X)) \leftarrow$ proved recently

$\hat{\ }$ rank

"Simple" example

$n=1$ Easy

$n=2$ $X = \mathbb{P}^1_{\mathbb{C}} = \overset{\mathbb{A}^1_{\mathbb{C}}}{\widetilde{\mathbb{C}}} \cup \{\infty\}$

$\pi_1^{\circ}(\mathbb{P}^1_{\mathbb{C}}) = 1$ easy

$g=2$ too complicated

Now try ramified case

Ramified Langlands

$\mathbb{P}^1_{\mathbb{C}} - \{x_1, x_2, x_3, x_4\}$

rank 2 $E$

$Conn_2(\mathbb{P}^1_{\mathbb{C}}, x_1, x_2, x_3, x_4)$ $(x_i \neq \infty)$

$= \{E \to \mathbb{P}^1_{\mathbb{C}}, \nabla : E \to E \otimes \Omega^1_X(x_1+x_2+x_3+x_4)\}$

$GL(2) \curvearrowright \mathbb{P}^1_{\mathbb{C}}$ $\overset{\infty}{O} \leftarrow \mathbb{A}^1_{\mathbb{C}}$

$E|_{\mathbb{A}^1_{\mathbb{C}}}$ is trivial $\nabla = d + A(z)dz$

$z$ coordinate on $\mathbb{A}^1_{\mathbb{C}}$

section of $E|_{\mathbb{A}^1}$ : $(f_1, f_2)$ $\quad f_i : \mathbb{C} \to \mathbb{C}$

$$\nabla \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} df_1 \\ df_2 \end{pmatrix} + \underset{\downarrow}{A(z)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dz$$

related

Peinlevé equation

$$\begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix}$$

near $x_i$ : $\quad A(z) = \frac{A_i}{z - z_i}$ + regular

Ex. Denote $\quad Conn'(\mathbb{P}^1, x_1, \cdots, x_4)$ the open subset of Conn where the bundle is trivial (nontrivial very complicated)

$$A(z) = \begin{cases} \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{z-z_3} + \frac{A_4}{z-z_4} \\ A_1 + A_2 + A_3 + A_4 = 0 \end{cases} \quad (*)$$

$\mathbb{C}^{12}$ vector space of parameter

$$\pi_1(\mathbb{P}^1 - \{x_1, x_2, x_3, x_4\}) \xrightarrow{\text{monodromy}} \mathbb{C}^2$$

analytic result

Can choose a trivilization of $E|_{\mathbb{A}^1}$

s.t. $\quad A_i = \begin{pmatrix} \alpha_i^+ & 0 \text{ or } 1 \\ 0 & \alpha_i^- \end{pmatrix}$

Fix some $\quad \alpha_i^{\pm} \in \mathbb{C}, \ i = 1, 2, 3, 4$

Look at $\quad M \subset Conn(\mathbb{P}^1, x_1, x_2, x_3, x_4)$

$$\forall \quad \underset{\substack{\| \\ A_i \in \operatorname{End}(E_{x_i})}}{\underline{\operatorname{Res}_{x_i} \nabla}} \sim \begin{pmatrix} \alpha_i^+ & 0 \\ 0 & \alpha_i^- \end{pmatrix}$$

Now $(*)$ $12 - 2 \cdot 4 = 4$ dim of parameters

Conditions (i) $\alpha_i^+ - \alpha_i^- \notin \mathbb{Z} \quad \forall i$

want a generic condition (ii) $\sum_{i=1}^{4} \alpha_i^\pm \notin \mathbb{Z}$ any $\pm$

$\sum \operatorname{tr} A^i$

(iii) $\sum_{i=1}^{4} \alpha_i^+ + \sum_{i=1}^{4} \alpha_i^- \in \mathbb{Z}$ "residue" condition ("Cauchy")

need to trivialize $A(z)$ in $(*)$

$/PGL(2)$

Finally dim 2

$\forall E, \nabla$

Thm. $\mathcal{M} \simeq M \times BG_m$ $\quad$ $\operatorname{Aut}(E, \nabla) \simeq \mathbb{C}^*$ "{scalars}"

where $M$ is a smooth quasi-projective surface

$M \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ $\quad$ $M \underset{\text{open}}{\subset} \overline{M}$ algebraic set

$\| \\ \left\{ \begin{array}{l} f_1 = f_m = 0 \\ g_1 \neq 0, \dots, g_m \neq 0 \end{array} \right\}$

$\tilde{P} = \{ (E, l_1, l_2, l_3, l_4) : E \xrightarrow[2]{} \mathbb{P}^1, l_i \subset E_{x_i}$ a line through the origin $\}$

$$\mathcal{M} \xrightarrow{\pi} \tilde{P}$$

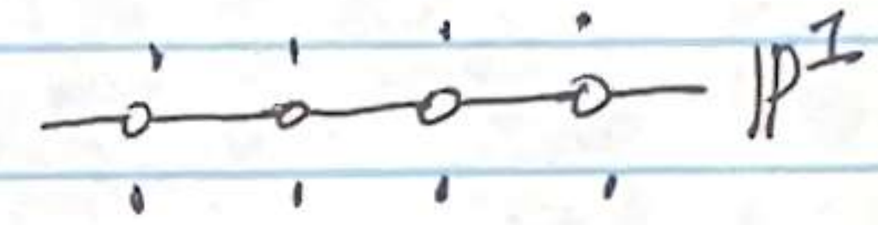
$(E, \nabla) \longmapsto (E,\ l_i = \alpha_i\text{-eigenspace of } \mathrm{Res}_{x_i} \nabla)$

$$\pi(\mathcal{M}) = \{(E, l_i) \text{ s.t. } \mathrm{Aut}(E, l_i) = \mathbb{C}^*\}$$

$$\shortparallel$$

$$\dot{P}$$

Prop. $P = \underline{P} \times BG_m$

$\underline{P} =$ 


$$QCoh^{(-1)}(\mathcal{M}) \simeq \mathcal{D}\text{-mod}^{\alpha_i^{\pm}}(P)$$

Moduli spaces

Classify up to isomorphism

Ex 1. Finite sets up to bijection $\mathbb{Z}_{\geq 0}$

Ex 2. Finite abelian groups / iso

Configuration space

Ex 3. $\mathbb{C}^n$

$\mathbb{P}^{n-1}_{\mathbb{C}} = \{\text{lines in } \mathbb{C}^n \text{ through origin}\}$

$Gr(n,d) = \{d\text{-dim linear subspaces in } \mathbb{C}^n\}$

$0 < d_1 < \cdots < d_\ell < n$

$Fl_{d_1 \cdots d_\ell}(\mathbb{C}^n) = \left\{ L_1 \subset L_2 \subset \cdots \subset L_\ell < \mathbb{C}^n \;\middle|\; \dim L_i = d_i \right\}$

flag varieties

$\binom{\alpha}{\beta} \in \mathbb{C}^2$

$\{\text{lines in } \mathbb{C}^2\} = \mathbb{P}^1_{\mathbb{C}} \ni \infty$

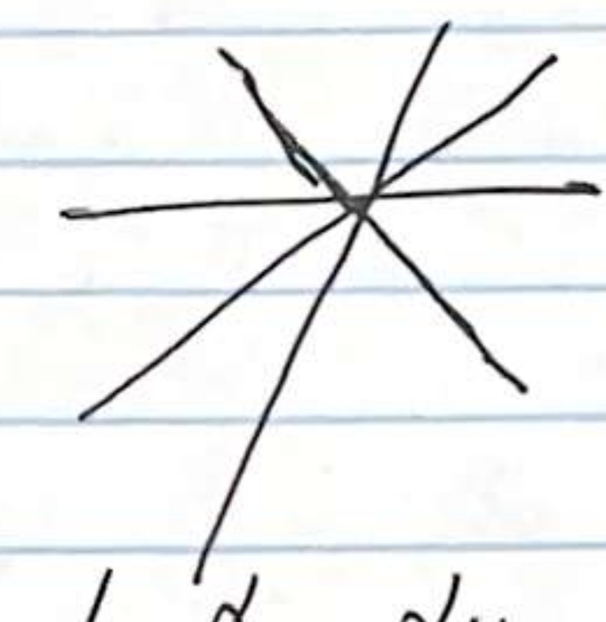
$GL(2) \curvearrowright \mathbb{P}^1_{\mathbb{C}}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \gamma \\ \sigma \end{pmatrix}$

moduli of lines in $\mathbb{P}^2$ is $\star$

moduli of 3 distinct lines is $*$

moduli of 4 distinct lines



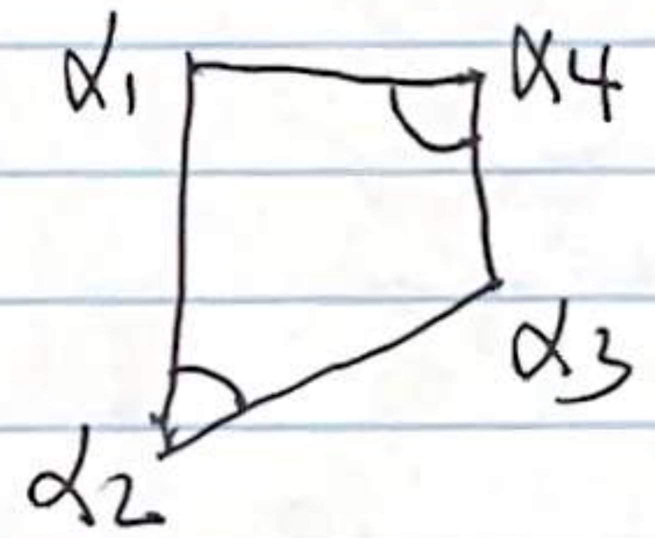
$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{P}^1_{\mathbb{C}}$

$0, 1, \infty$

$$\frac{\alpha_1 - \alpha_2}{\alpha_3 - \alpha_2} \Big/ \frac{\alpha_1 - \alpha_4}{\alpha_4 - \alpha_3}$$

double ~~radial~~ ratio

up to this




In $0, 1, \infty$ case

$$= \frac{1}{\alpha_4} \neq 0, 1, \infty$$

moduli of 4 distinct points on $\mathbb{P}^1$ is

$$\mathbb{P}^1 - \{0, 1, \infty\}$$

Vector bundles in AG

$B$ variety

Def. V.B. of rank $r$ over $B$

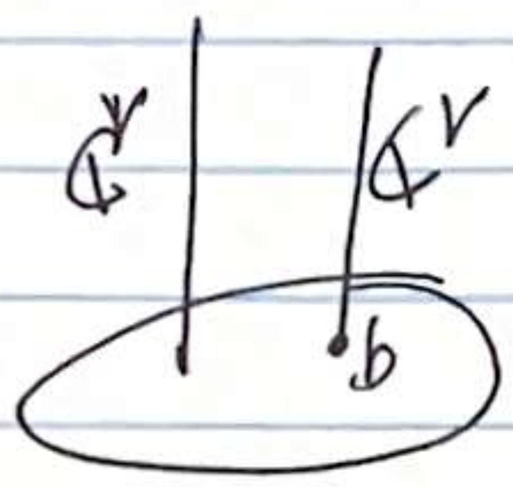
(i) a variety $E$ (total space)

(ii) $\pi : E \to B$

(iii) $\forall\, b \in B$ a vector space of dim $r$

structure on $\pi^{-1}(b)$

$E = B \times \mathbb{C}^n$ trivial bundle




(iv) $\forall\, b \in B$ $\exists$ open $U \subset B$, $b \in U$

and an iso $\pi^{-1}(U) \simeq U \times \mathbb{C}^r$

compatible with (iii)

$\mathrm{Bun}_r(X)$ moduli space of ~~line~~ rank $r$ bundles on algebraic curves

$r = 1$ line bundle

$$\mathbb{P}^n_{\mathbb{C}} = \{ l \subset \mathbb{C}^{n+1} \}$$

$$\mathcal{L} = \{ (l, x) \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} : x \in l \}$$

$$\begin{tikzcd} \mathcal{L} \arrow[d, "\pi"'] \\ l \in \mathbb{P}^n_{\mathbb{C}} \end{tikzcd}$$

$\pi^{-1}(l) = l$ 1-dim v.s.

$\mathcal{L} \to \mathbb{P}^n_{\mathbb{C}}$ "tautological" line bundle

$\mathcal{L} \subset \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$

$$\begin{tikzcd} E' \arrow[r, phantom, "\subset"] \arrow[dr] & E \arrow[d] \\ & B \end{tikzcd}$$

$$\begin{tikzcd} f^*\mathcal{L} \arrow[d] & \mathcal{L} \arrow[d] \\ X \arrow[r, "f"'] & \mathbb{P}^n_{\mathbb{C}} \end{tikzcd}$$

In general

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

$$E \times_B B' = \{ (e, b') \in E \times B' : \pi(e) = f(b') \}$$

$$\begin{tikzcd}
X \times \cancel{\mathbb{P}} \mathbb{C}^{n+1} \arrow[d] & \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1} \arrow[d] \\
X \arrow[r] & \mathbb{P}^n_{\mathbb{C}}
\end{tikzcd}$$

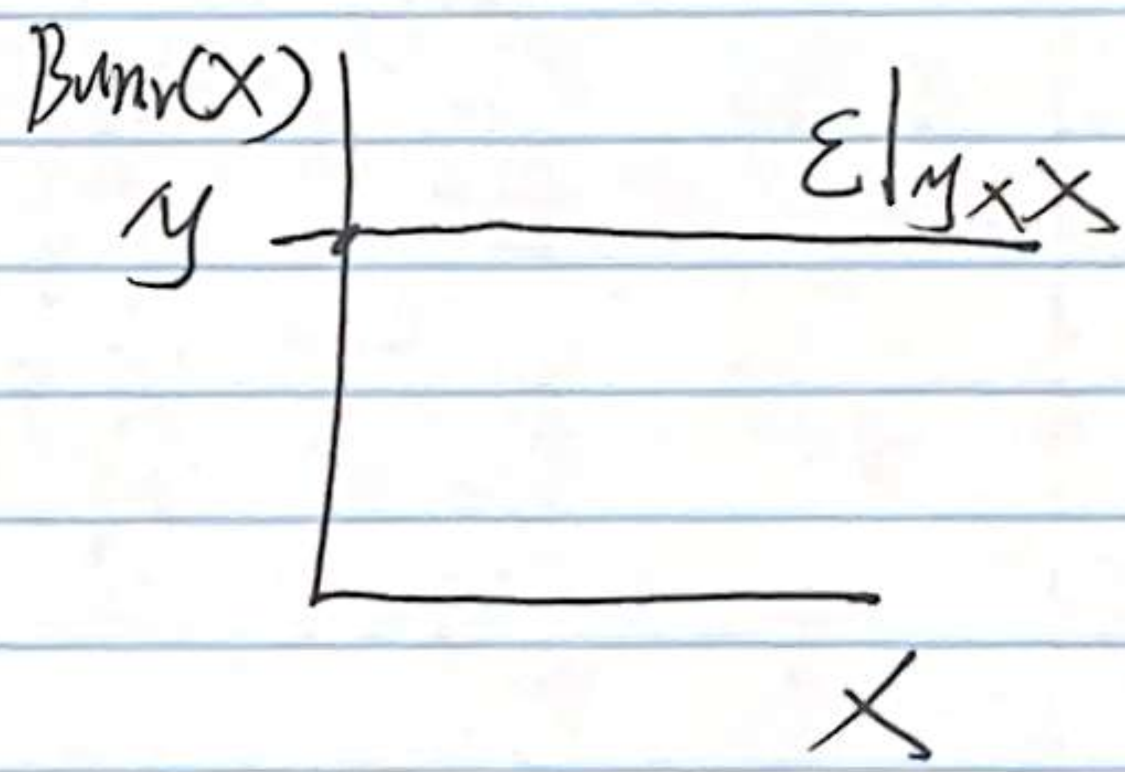
Summary

$$\{ X \xrightarrow{f} \mathbb{P}^n_{\mathbb{C}} \} \rightsquigarrow \left\{ \begin{tikzcd}
L \arrow[r, hook, "\text{embedding}"] \arrow[d] & X \times \mathbb{C}^{n+1} \arrow[dl] \\
X &
\end{tikzcd} \right\}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[d] \\
\mathrm{Bun}_r(X) \times X
\end{tikzcd}$$

$\Downarrow$

$y$




$$y \rightsquigarrow (E_y \xrightarrow{r} X)$$

$$\mathcal{E}|_{\{y\} \times X} \simeq E_y$$

$$\begin{tikzcd}
E \arrow[d, "r"] \\
\cancel{X} S \times X
\end{tikzcd} \rightsquigarrow \quad \begin{tikzcd}
E \arrow[d, phantom, "\wr"] & \\
(\mathrm{Id}_f \times X)^* \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
S \times X \arrow[r, "\mathrm{Id}_f \times X"] & \mathrm{Bun}_r(X) \times X
\end{tikzcd}$$

$\exists!\ f : S \to \mathrm{Bun}_r(X)$

$\mathrm{Id}_f \times X : S \times X \to \mathrm{Bun}_r(X) \times X$

"Algebraic curves"

projective smooth connected

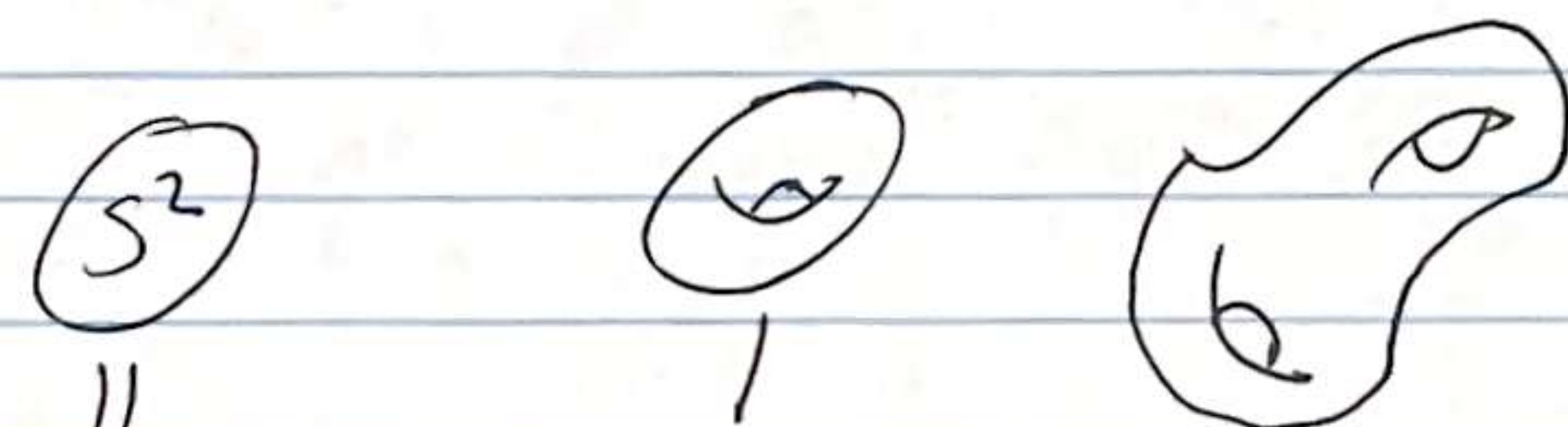
$\exists\ X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$

$\underset{\text{closed}}{V \subset X} \Rightarrow V = X$ or $|V| < \infty$

$X \rightsquigarrow$ 2 dim surface (real)

compact, orientable

$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$




$\| \qquad\qquad |$

$\mathbb{C}\mathbb{P}^2$ $\qquad$ $y^2 z = x^3 + a x^2 z + b z^3$

elliptic curve

$Bun_r(X)$ somewhat boring $\qquad\qquad$ $Bun_r(X)$ complicated

parabolic bundles

Line bundles on $\mathbb{P}^n$

$\mathcal{L} = \mathcal{O}(-1)$ tautological line bundle

$\mathcal{O}(1) = \mathcal{O}(-1)^\vee$ $\qquad\qquad$ $\mathcal{O}(0) = \mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}$ trivial

$\mathcal{O}(n) = \overbrace{\mathcal{O}(1) \otimes \cdots \otimes \mathcal{O}(1)}^{n}$

Thm. $\mathcal{L}$ is a line bundle on $\mathbb{P}^n$

$\exists!$ $n$: $\mathcal{L} \simeq \mathcal{O}(n)$.

Thm. Let $E \to \mathbb{P}^1_{\mathbb{C}}$ be a vector bundle of rank $r$.

Then $\exists$ $n_1, \cdots, n_r$ (unique up to permutation)

$$E \simeq \mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_r)$$

Ex. $r=2$ $\quad E \simeq \mathcal{O}(n) \oplus \mathcal{O}(m)$

$x_1, \cdots, x_{m} \in \mathbb{P}^1_{\mathbb{C}}$

A parabolic bundle of rank 2 type $(x_1, \cdots, x_m)$

consists of (i) $E \xrightarrow{2} \mathbb{P}^1_{\mathbb{C}}$

(ii) $\forall i = 1, \cdots, m$

a line $l_i \in E_{x_i} \simeq \mathbb{C}^2$

$Par_2(\mathbb{P}^1_{\mathbb{C}}, x_1, \cdots, x_e)$ parametrizing such p.b.

open subspace ptw. consisting $(E, l_1, \cdots, l_n)$

where $E$ is a trivial v.b. $(E \simeq \mathbb{P}^1 \times \mathbb{C}^2)$

Pick $E \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{C}^2$

$$\begin{array}{ccc} E_{x_i} & \xrightarrow{\sim} & \mathbb{C}^2 \\ \cup & & \cup \\ l_i & & \end{array}$$

$(E, l_1, \cdots, l_m) \rightsquigarrow (l_1, \cdots, l_m \subset \mathbb{C}^2)$

~~If~~ Choose a different trivilization

amounts to acting by $GL_2$

$m=4$ $\quad P^{\text{very triv}} \subset P^{\text{triv}}$

correspond to $(l_1, \cdots, l_4)$ $l_i \neq l_j$ if $i \neq j$

$P^{\text{very triv}} \simeq \mathbb{P}^1 - \{0, 1, \infty\} = \mathbb{A}^1 - \{0, 1\}$

$\cap$ open

$Par_2(\mathbb{P}^1_{\mathbb{C}}, x_1, \cdots, x_4)$

Vector bundles

$M$ - smooth manifold

$E \to M$

$C^\infty(E) = \{ \overset{s}{\cancel{\theta}} : M \to E,\ \pi \circ \cancel{E} = id_M \}$

$f \in C^\infty(M)$ - module

$(fs)(m) = f(m)s(m)$

$\mathrm{Vect}(M) \to C^\infty(E)$-mod

$k = \bar{k}$

$\underset{\text{closed}}{X \subset k^n}$

$\mathrm{Vect}(X) \hookrightarrow k[X]$-module

$X = k^1 \qquad k[X] = k[t]$

$\mathrm{Vect}(X) \hookrightarrow k[\overset{t}{\cancel{X}}]\text{-mod}^{f.g.}$

$\quad\quad\quad\quad\quad\quad\quad \Psi$

$\quad\quad\quad\quad\quad\quad\quad M$

$M \simeq k[t]^{\oplus r} \oplus \bigoplus_{i=1}^{n} k[t]/(t-a_i)^{n_i}$

$R$ - PID $\qquad M \simeq \oplus M_i \qquad M_i \simeq R$ or $M_i \simeq R/m_i^{n_i}$

$M$ is a vector bundle $\Longleftrightarrow$ $M \simeq k[t]^{\oplus r}$

$$M \simeq X \times k^r$$

(if not, $(t-a_i)^{n_i} s = 0$
$\uparrow$ torsion
$s=0$ for $t \neq 0$ )

$X \subset k^n$ $X$ is a smooth connected curve

$k[X]$ – Dedekind ring

$\mathbb{P}^1$ $k[\mathbb{P}^1_k] = k$ sections of $E$ over $U$

$E \mapsto (\forall\, U \underset{\text{open}}{\subset} \mathbb{P}^1_k,\ \Gamma(X, U))$

$k[U]$-module

sheaf of $\mathcal{O}_{\mathbb{P}^1}$-module

$\mathbb{P}^1_k \supset k = \mathbb{P}^1 - \infty$
$\cup$

$k^1 = \mathbb{P}^1 - 0$

$E \to \mathbb{P}^1$

$E|_{\mathbb{P}^1-\infty} \to k[t]$-module $E_1 \xrightarrow{\sim} k[t]^{\oplus r}$

$E|_{\mathbb{P}^1-0} \to k[s]$-module $E_2 \xrightarrow{\sim} k[s]^{\oplus r}$

intersection $(\mathbb{P}^1-\infty) \cap (\mathbb{P}^1 - 0) = k - 0$

need to glue $\mathcal{E}_1|_{k-0} \simeq \mathcal{E}_2|_{k-0}$

$$\mathcal{E}_1|_{k-0} = \mathcal{E}_1 \otimes_{k[t]} k[t,t^{-1}] = (\mathcal{E}_1)_t$$

Now $k[t,t^{-1}]^{\oplus r} \xrightarrow{\sim} k[t,t^{-1}]^{\oplus r}$

parametrized by $\phi \in GL_r(k[t,t^{-1}])$

$$Hom(k[t,t^{-1}]^{\oplus r}, k[t,t^{-1}]^{\oplus r})$$
$$= \bigoplus_{i,j}^{r} Hom(k[t,t^{-1}], k[t,t^{-1}])$$

$$Hom_R(R,R) = R \Rightarrow Hom(R^{\oplus r}, R^{\oplus r}) = gl_r(R)$$

$$\mathcal{L} \to \mathbb{P}^1$$
$$\uparrow$$
$$\phi \in gl_1(k[t,t^{-1}]) = k[t,t^{-1}]^* = \{at^n \mid a \in k\}$$

$\mathcal{O}_{\mathbb{P}^1}(n) =$ trivial on $\mathbb{P}^1 - \infty$ $\underbrace{\qquad}_{t^n}$ trivial on $\mathbb{P}^1 - 0$

glue

$$Bun_r(\mathbb{P}^1)_{set} = {}_{GL_r(k[t^{-1}])}\backslash GL_r(k[t,t^{-1}]) / GL_r(k[t])$$

$H \subset G \supset K$ $\{HgK \mid g \in G\} = H\backslash G/K$

double coset

$z \in k$

$$A(t) = \begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix} \in GL_2(\mathbb{C}[t, t^{-1}])$$

$$\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} t^{-1} & -z \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A(z) \rightsquigarrow E_z$

$z = 0$ : $E_t \approx \mathcal{O}(1) \oplus \mathcal{O}(-1)$ $\qquad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

$z \neq 0$ : $E_t \approx \mathcal{O} \oplus \mathcal{O}$

$f, g \in \mathbb{C}[t]$

$$\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} tf + g \\ t^{-1} g \end{pmatrix}$$

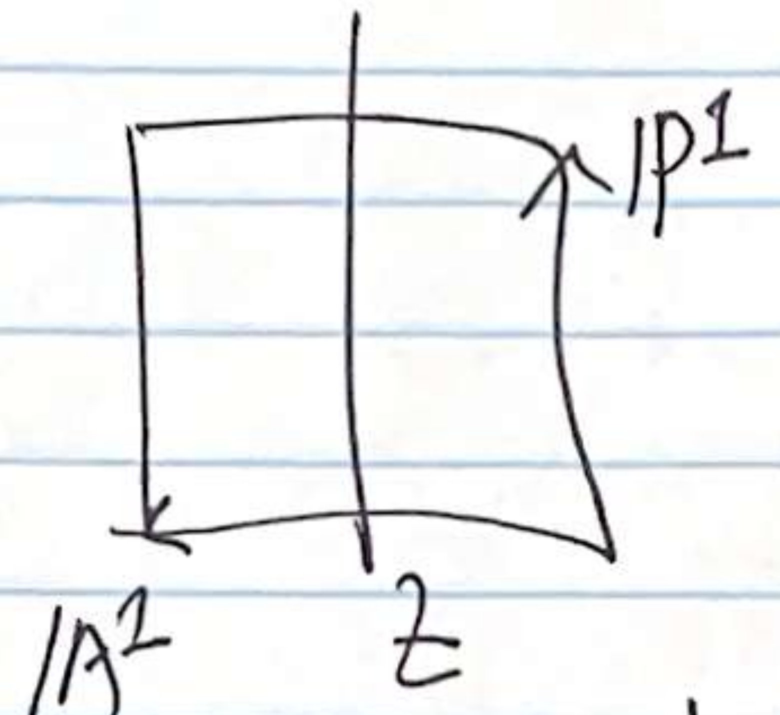
$$\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}\overbrace{\begin{pmatrix} 0 & -1 \\ 1 & tz^{-1} \end{pmatrix}}^{=A_+} \in GL_2(k[t^{-1}])$$

$$= \begin{pmatrix} z & 0 \\ t^{-1} & z^{-1} \end{pmatrix}^{=A_-}$$

$$\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix} = A_+^{-1} A_-$$

$$E \to \mathbb{P}^1 \times \mathbb{A}^1 = ((\underset{t}{\mathbb{P}^1 - \infty}) \times \underset{z}{\mathbb{A}^1}) \sqcup ((\mathbb{P}^1 - 0) \times \mathbb{A}^1)$$

glue by $\begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix}$




$$E|_{\mathbb{P}^1 \times z} = E_z = \begin{cases} \mathcal{O} \oplus \mathcal{O}, & z \neq 0 \\ \mathcal{O}(1) \oplus \mathcal{O}(-1), & z = 0 \end{cases}$$

family on $\mathbb{P}^1$ parametrized by $\mathbb{A}^1$

$$E_1 \xrightarrow{2} \mathbb{P}^1 \xleftarrow{2} E_2$$

$$E_1 \simeq \mathcal{O}(m) \oplus \mathcal{O}(n)$$

$$E_2 \simeq \mathcal{O}(m') \oplus \mathcal{O}(n')$$

$$\exists\ S \ni \overbrace{s_1, s_2}^{\text{connected}}$$

$$E \to \mathbb{P}^1 \times S$$

$$E|_{\mathbb{P}^1 \times s_1} \simeq E_1$$

2

$$E|_{\mathbb{P}^1 \times s_2} \simeq E_2$$

$\mathrm{Bun}_2(\mathbb{P}^1)$ Answer: $\iff m+n = m'+n'$

$$E \simeq \mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_k)$$

$$\deg E = n_1 + \cdots + n_k$$

$\mathrm{Bun}_{2,d}(\mathbb{P}^2) \subset \mathrm{Bun}_2(\mathbb{P}^2)$

$\uparrow$
v.b. of deg $d$

connected component $\quad \mathrm{Bun}_2(\mathbb{P}^1) = \bigsqcup_d \mathrm{Bun}_{2,d}(\mathbb{P}^1)$

decomposition

$\deg E_1 = \deg E_2$

~~Hom$(E_1, E_2) \neq 0$~~ $\{E_1 \hookrightarrow E_2\} \neq \phi$

$\Rightarrow E_1 \simeq E_2$

$X$ smooth projective curve $/k$

$E$ vector bundle $\rightsquigarrow \deg E$

$\mathrm{Bun}_r(X) = \bigsqcup_d \mathrm{Bun}_{r,d}(X)$

$x_1, x_2, x_3, x_4 \in \mathbb{P}^1$

$\mathrm{Par}_2(X, x_1, x_2, x_3, x_4) = \{E \underset{2}{\to} \mathbb{P}^1,\ l_i \in E_{x_i}\}$

$\cup$

$\mathrm{Par}_{2,-1}(X, x_1, \cdots, x_4) \leftarrow$ of infinite type

$\uparrow$
$\deg E = -1$

$\{\phi : E \to E \mid \phi(l_i) = l_i\}$
$\quad ||$

$\cup$

$\mathrm{Par}'_{2,-1}(X, x_1, \cdots, x_4) = \{\mathrm{Aut}(E, l_i) \simeq k^\times\}$

Recall.

$\mathrm{End}(\mathcal{O} \oplus \mathcal{O}(10)) = \mathrm{Hom}(\mathcal{O} \oplus \mathcal{O}(10), \mathcal{O} \oplus \mathcal{O}(10))$

$$= \underbrace{\mathrm{Hom}(\mathcal{O},\mathcal{O})}_{\substack{\cong \\ k}} \oplus \underbrace{\mathrm{Hom}(\mathcal{O},\mathcal{O}(10))}_{\substack{\cong \\ k^{11}}} \oplus \underbrace{\mathrm{Hom}(\mathcal{O}(10),\mathcal{O})}_{0} \oplus \underbrace{\mathrm{Hom}(\mathcal{O}(10),\mathcal{O}(10))}_{\substack{\cong \\ k}}$$

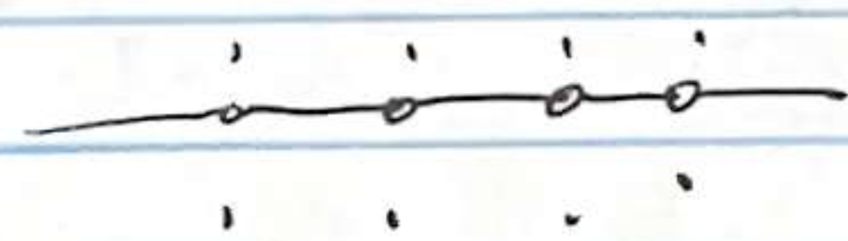
Prop. $(E, l_i) \in \mathrm{Par}'_{2,-1}$

$$\Rightarrow E \cong \mathcal{O} \oplus \mathcal{O}(-1)$$

Rigidification

$$\mathrm{Par}''_{2,-1} = \{(E, l_i) \in \mathrm{Par}'_{2,-1},\ \overset{s \neq 0}{s \in \Gamma(\mathbb{P}^1, E)}\}$$

Thm. $\simeq \quad \mathbb{P}^1 \underset{\mathbb{P}^1 - \{x_1, x_2, x_3, x_4\}}{\sqcup} \mathbb{P}^1$



Vector bundles on "varieties"

$E/X \supset_{\text{open}} \mathcal{U}$

$\underline{E}(\mathcal{U}) = \Gamma(\mathcal{U}, E)$

identify vector bundles with sheaves of sections

$\mathrm{Vect}(X) \subset \mathrm{Coh}(X)$

$X = \operatorname{Spec} R$ ($R$ noetherian)

$$\begin{array}{ccc} \mathrm{Coh}(X) & \simeq & \mathrm{Mod}^{f.g.}(R) \\ \cup & & \cup \\ \mathrm{Vect}(X) & & \mathrm{Mod}^{\text{proj.}, f.g.}(R) \end{array}$$

"Linear algebra" of v.b.

$(E \oplus F)_x = E_x \oplus F_x$

$E \otimes F$

$E^\vee \qquad (E \otimes F)^\vee = E^\vee \otimes F^\vee$

$\mathcal{H}om(E, F) = E^\vee \otimes F$

↑
also vector bundle

$\mathrm{Hom}(E, F) \leftarrow$ vector space

$\Gamma(X, \mathcal{H}om(E, F)) \simeq \mathrm{Hom}(E, F)$

$\mathrm{Hom}(\underset{X \times \mathbb{C}}{\underset{\parallel}{\mathcal{O}_X}}, E) = \Gamma(X, E)$

$\mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(m)) \Leftarrow$ on $\mathbb{P}^1_{\mathbb{C}}$

$n \geq 0$ $\quad \mathcal{O}(n)(U) = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(U) & \text{if } \infty \notin U \\ \text{functions on } U \text{ having a pole of order} \leq n & \text{if } \infty \in U \end{cases}$

$$\Gamma(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \text{polynomials of degree} \leq n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(m)) &= \Gamma(\mathbb{P}^1, \mathcal{H}om(\mathcal{O}(n), \mathcal{O}(m))) \\ &= \Gamma(\mathbb{P}^1, \mathcal{O}(n)^\vee \otimes \mathcal{O}(m)) \\ &= \Gamma(\mathbb{P}^1, \mathcal{O}(m-n)) \\ &= \begin{cases} \text{poly of degree} \leq m-n & \text{if } m-n \geq 0 \\ 0 & \text{if } m-n < 0 \end{cases} \end{aligned}$$

the above discussion also works for $\mathbb{P}^N_k$

$$\mathrm{End}(\mathcal{O}(n) \oplus \mathcal{O}(m)) \underset{n \leq m}{=} \mathrm{Hom}(\mathcal{O}(n) \oplus \mathcal{O}(m), \mathcal{O}(n) \oplus \mathcal{O}(m))$$

$$\psi$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$a \in \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n)) = \mathbb{C}$ $\qquad d \in \mathbb{C}$

$b \in \mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n)) = 0$ $\qquad c \in \mathbb{C}^{m-n+1}$

....

End $E$ - vector space ($k$-algebra)

$X$ projective $/k$ $\quad$ ~~$E$~~ ~~coherent~~ $E \in Coh(X)$

$$\dim \Gamma_{k}(X, E) < \infty$$

$$End(E) = \Gamma(X, E^\vee \otimes E)$$

$\cup$

Aut $E$ $\qquad$ $A \in$ Aut iff $\quad a \neq 0 \neq d$

$\dim$ Aut $E$ = $\dim$ End $E$

$$\dim Aut(\mathcal{O}(n) \oplus \mathcal{O}(m)) = m-n+3$$

$\cup$ $n<m$

$k^* \ni \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = A$

$k \subset$ End $E$

$k^* \subset$ Aut $E$

Lemma. $E = \mathcal{O}(n) \oplus \mathcal{O}(m)$ $(n<m)$ has a unique subbundle isomorphic to $\mathcal{O}(m)$.

Pf. $\quad \mathcal{O}(m) \subset \mathcal{O}(m) \oplus \mathcal{O}(n)$

$$L \overset{i}{\hookrightarrow} \mathcal{O}(m) \oplus \mathcal{O}(n) \qquad L \cong \mathcal{O}(m)$$

$$i \in Hom(L, \mathcal{O}(m) \oplus \mathcal{O}(n))$$

$$\simeq Hom(\mathcal{O}(m), \mathcal{O}(m) \oplus \mathcal{O}(n)) \simeq \mathbb{C}$$

$\exists\ \infty$ subbundles $\approx \mathcal{O}(n)$

Cor. $\phi \in \mathrm{Aut}(\mathcal{O}(m) \oplus \mathcal{O}(n))$

$$\Rightarrow \phi(\mathcal{O}(m)) = \mathcal{O}(m)$$

$\mathcal{D} = x_1 + x_2 + x_3 + x_4 \subset \mathbb{P}^2$

$x_i \neq x_j \quad x_i \neq \infty$

$\mathrm{Par}_2(\mathbb{P}^1, \mathcal{D}) = \{ E \xrightarrow[2]{} \mathbb{P}^1,\ l_i \in E_{x_i} \}$

$\tilde{E} = (E = \mathcal{O}(n) \oplus \mathcal{O}(m) \mid l_i = \mathcal{O}(m)_{x_i})$

Claim. $\mathrm{End}(\tilde{E}) = \mathrm{End}(E)$

$\phi : \tilde{E} \to \tilde{E} \quad \phi(l_i) = l_i$

dim at least 4

$\mathrm{Par}_2^1(\mathbb{P}^1, \mathcal{D}) = \{(E, l_i),\ \mathrm{Aut}(E, l_i) = \mathbb{C}^*\}$

Assume $(E, l_i) \in \mathrm{Par}_2^1(\mathbb{P}^2, \mathcal{D})$

$\deg E = 1 \Rightarrow E \approx \mathcal{O} \oplus \mathcal{O}(1)$

possible $\mathcal{O}(-n) \oplus \mathcal{O}(1+n)$ but not

$\mathcal{O}(-2) \oplus \mathcal{O}(2)$

Assume $E \approx \mathcal{O}(-1) \oplus \mathcal{O}(2)$

want to prove $\mathrm{Aut}(E, l_i) \neq k^*$

$$\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \qquad \begin{array}{l} d \in \mathbb{C}^* \\ c \quad \text{poly of degree} \leq 3 \end{array}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} = \overset{\varphi}{\cancel{\Phi}} \in \mathrm{End}(E)$$

$\mathcal{O}(m)|_{\mathbb{A}^1} \simeq \mathcal{O}$

$\mathcal{O}(m)|_x = \mathbb{C}$

$\mathcal{O}(m) \oplus \mathcal{O}(n)|_x = \mathbb{C} \oplus \mathbb{C}$

$\varphi \cancel{\Phi}(l_i) = l_i$ ?

$l_i = \mathbb{C}(1, \alpha_i)$

$l_i = \mathbb{C}(0, 1) = \mathcal{O}(m)_{x_i}$

$$\varphi(1, \alpha_i) = \begin{pmatrix} 0 & 0 \\ c(x_i) & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \alpha_i \end{pmatrix} = \begin{pmatrix} 0 & \overset{c(x_i)}{\cancel{a}} + \alpha_i \end{pmatrix}$$

$$= \lambda(1, \alpha_i)$$

$$= (0, 0)$$

degree 3 $\Rightarrow c(x_i) + \alpha_i = 0$

$\mathrm{End}(E_{\ell}, l_i) = (\varphi : \varphi(l_i) \subset \cancel{\Phi} l_i)$

Now $\mathrm{Par}^1_{2,1}(\mathbb{P}^1, D) = \{\mathcal{O} \oplus \mathcal{O}(1), l_i\} \sim 1$ dim same calculation with $\mathcal{O} \oplus \mathcal{O}(1)$?

- 2 of $l_i$ cannot lie on $\mathcal{O}(1)$

  $(x_i, \alpha_i)$ cannot lie on a single line

  $l_i \subset (\mathcal{O} \oplus \mathcal{O}(1))_{x_i} = \mathbb{C} \oplus \mathbb{C}$

  $l_i \in \mathbb{P}^1$

  $\{l_1, l_2, l_3, l_4\} \in (\mathbb{P}^1)^4 \circlearrowleft \overbrace{\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(1))}^{4-1=3} / \mathbb{C}^*$

  at most two of $l_i = \infty$

Thm. $\widetilde{\mathrm{Par}}_{2,1}(\mathbb{P}^1, D) = \{(E, l_i, s) : (E, l_i) \in \mathrm{Par}'_{2,1}(\mathbb{P}^1, D),$

$s : \mathcal{O}(1) \hookrightarrow E \}$

rigidification $\approx \mathbb{P}^1 \sqcup_{\mathbb{P}^1 - D} \mathbb{P}^1$

$(E, l_i) \to E_i \qquad x_i \in \mathbb{P}^1$

$E_i(U) = \{ s \in E(U) : \underset{\substack{\uparrow \\ E_{x_i}}}{s(x_i)} \in l_i \}$

$X/k$ 1-dim of finite type

either $X \simeq \operatorname{Spec} R \underset{\text{closed}}{\hookrightarrow} \mathbb{A}^n_k$

or $X \underset{\text{closed}}{\subseteq} \mathbb{P}^2_k$

(If $X \simeq \operatorname{Spec} R$ smooth ⟸ local question

$R$ is a Dedekind ring — start point

$\mathrm{Coh}(X) \simeq \mathrm{Mod}^{f.g.}(R)$ — but global one will be more complicated

$X/k$ smooth $\mathcal{F} \in \mathrm{Coh}(X)$

$$\mathcal{F}^{pre}_{tors}(U) = \left\{ s \in \mathcal{F}(U) : \exists f \in \mathcal{F}(U),\ f \neq 0 \atop fs = 0 \right\}$$

↑ only presheaf

$$\mathcal{F}_{tors}(U) = \left\{ s \in \mathcal{F}(U) : \exists U = \bigcup U_i \atop \forall i\ \exists f_i \in \mathcal{F}(U_i)\ f_i s|_{U_i} = 0,\ f_i \neq 0 \right\}$$

$\mathcal{F}_{tors} \subseteq \mathcal{F}$ $\mathrm{Coh}(X)$ is abelian

$\mathcal{F}_{vect} := \mathcal{F}/\mathcal{F}_{tors}$ – torsion free i.e. $(\mathcal{F}/\mathcal{F}_{tors})_{tors} = 0$

($\Leftrightarrow$ locally free)

local statement

Pf. May assume $X \simeq \operatorname{Spec} R$

$$M = M_{\text{locfree}} \oplus \underbrace{\bigoplus_i R/p_i^{n_i}}_{M_{\text{tors}}}$$

$$0 \to F_{\text{tors}} \to F \to F_{\text{vect}} \to 0 \quad \text{splits}$$

$$F \simeq F_{\text{tors}} \oplus F_{\text{vect}} \quad (\text{not canonical})$$

$$(\text{parameterized by } \operatorname{Hom}(F_{\text{vect}}, F_{\text{tors}}))$$

Torsion sheaves

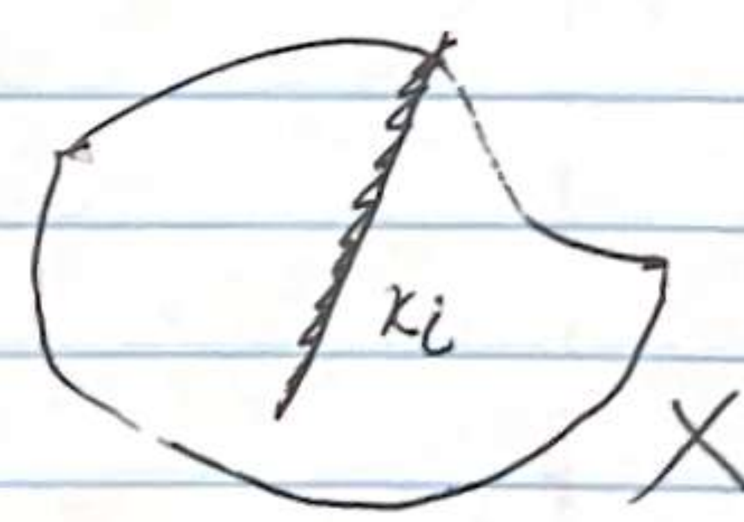
$p_i \longleftrightarrow x_i \in X$ (closed point)

$R/p_i^{n_i} \longleftrightarrow \mathcal{O}_{n_i x_i}$

set theoretically supported at $x_i$

i.e. $\forall\, U \subset X - x_i$

structural sheaf $\mathcal{O}_{n_i x_i}(U) = 0$




$n_i x_i$ closed subscheme of $X$

$n_i = 1$ $\mathcal{O}_{x_i}$ skyscraper sheaf at $x_i$

$r(F) =$ rank of $F_{\text{vect}}$
$\in \mathbb{Z}$

Now $X$ is projective

~~$\deg(\mathcal{O}_{n_i x_i})$~~

$[k(x_i):k]$ $\quad n_i > 0$

$\deg(\bigoplus_i \mathcal{O}_{n_i x_i}) = \sum n_i \boxed{\deg x_i}$ $\quad$ can have $\mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2}$

$(= \sum n_i$ if $\bar{k} = k)$

$F = \mathcal{L} = \mathcal{O}_X(D)$ v.b. of ~~deg~~ rank 1. $\quad D = n_i x_i \quad x_i \neq x_j$

$\deg \mathcal{L} = \sum n_i [k(x_i):k]$

Now $\deg F_{vect} = \deg \underbrace{\Lambda^r F_{vect}}$

$r =$ rank $F_{vect}$ $\quad$ determinant of $F_{vect}$

Now $\deg F = \deg F_{tors} + \deg F_{vect} \in \mathbb{Z}$

$0 \to F_1 \to F_2 \to F_3 \to 0$ s.e.s.

$\Rightarrow \deg F_2 = \deg F_1 + \deg F_3$

$F \in Coh(X)$ is a vector bundle $\Leftrightarrow F_{tors} = 0$

If $F' \subset F$, $F$ is a v.b. $\Rightarrow F'$ v.b.

Ex. $X = \operatorname{Spec} k[t]$

$\underbrace{(t-a) \cdot k[t]}_{\mathcal{L}_1} \subset \underbrace{k[t]}_{\mathcal{L}_2}$

line bundle

$\ell_2/\ell_1$ supported at a

sky-scraper sheaf

$x \in X$

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to \mathcal{O}_x \to 0$$

$\mathcal{E}$ is ~~too~~ locally free  supported at $x$

$i_x : x \hookrightarrow X$

$$0 \to \mathcal{E}(-x) \to \mathcal{E} \to \mathcal{E}_x \to 0 \quad \mathcal{E}_x := i_{x,*} i_x^* \mathcal{E}$$

$$\Gamma(U, \mathcal{E}(-x)) = \{ s \in \mathcal{E}(U),\ s(x) = 0 \}$$

Find $\mathcal{F}$ s.t. $\mathcal{E}(-x) \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ ~~rk $\mathcal{E}$ = 2~~ $(*)$

~~Ex.~~ In general $\mathcal{E}_x$ $k(x)$-vector space of dim $r = \operatorname{rank} \mathcal{E}$

$V \subset \mathcal{E}_x$ ~~$\mathcal{E}_V(U) := \{ s \in \mathcal{E}(U)$~~

$$\mathcal{E}_{x,V}(U) := \{ s \in \mathcal{E}(U) : s_x \in V \}$$

$\mathcal{E}_{x,V} \subset \mathcal{E}$ subsheaf

If $V = \mathcal{E}_x$, $\mathcal{E}_{x,V} = \mathcal{E}$

$V = 0$, $\mathcal{E}_{x,V} = \mathcal{E}(-x)$

Ex. Every $\mathcal{F}$ satisfying $(*)$ is $= \mathcal{E}_{x,V}$ for $!V$

Digression $k = \mathbb{F}_q$

$G$ ~~$\mathbb{Q}$~~$_{fin}$ [Isoclasses of vector bundles on $X$, rank $r$]

$\circlearrowright \qquad x \in X \qquad 1 \le l \le r-1$

$H_{x,l}$ Hecke operator

$$\underbrace{(H_{x,l} f)}_{\text{commute}}(\mathcal{E}) = \sum_{\substack{V \subset \mathcal{E}_x \\ \dim V = l}} f(\mathcal{E}_{x,V})$$

Look for common eigen-functions

$\mathrm{Par}_{2,d}(\mathbb{P}^1_{\mathbb{C}}, x_1, \cdots, x_4) \supset \mathrm{Par}^1_{2,d} \qquad \{\varphi : \mathcal{E} \to \mathcal{E},\ \varphi(l_i) \subset l_i\}$

$\quad \cup \qquad\qquad\qquad\qquad \uparrow$

$(\mathcal{E}, l_i \subset \mathcal{E}_{x_i}) \qquad \mathrm{End}(\mathcal{E}, l_i \subset \mathcal{E}_{x_i}) = \mathbb{C}$

Last time. $\quad (\mathcal{O}(m) \oplus \mathcal{O}(n), l_i) \in \mathrm{Par}^1_2$

$$|m-n| \le 2$$

If $d$ is ~~even~~ odd : $\mathcal{E} = \mathcal{O}(\frac{d-1}{2}) \oplus \mathcal{O}(\frac{d+1}{2})$

Ex. $\quad d=0 \quad \mathcal{O} \oplus \mathcal{O},\ \mathcal{O}(-1) \oplus \mathcal{O}(1)$

$\quad\quad d=1 \quad \mathcal{O} \oplus \mathcal{O}(1)$

$(\mathcal{E}, l_i \subset \mathcal{E}_i) \otimes \mathcal{O}(n) \qquad \mathcal{O}(n)_{x_i} \simeq \mathbb{C} \quad x_i \neq \infty$

~~$\mathcal{E}(n)$~~ $\qquad \mathcal{E}(n)_{x_i} = \mathcal{E}$

$\qquad\qquad\qquad\qquad \cup \qquad \cup$

$\mathrm{Par}_{2,d} \xrightarrow{\sim} \mathrm{Par}_{2,d+2n} \qquad l_i \qquad l_i$

$$(\mathcal{O}(a) \oplus \mathcal{O}(b) \mapsto \mathcal{O}(a+n) \oplus \mathcal{O}(b+n))$$

$$\overset{\cup}{\mathrm{Par}^1_{2,d}} \xrightarrow{\sim} \mathrm{Par}^1_{2,d+2n}$$

Lemma. $\forall d, d'$

$$\mathrm{Par}^1_{2,d} \simeq \mathrm{Par}^1_{2,d'}$$

Pf. construct isomorphism

$$\mathrm{Par}^1_{2,d} \xrightarrow{L_i} \mathrm{Par}^1_{2,d-1} \qquad \text{choose } x_i$$

$$(\mathcal{E}, l_i) \mapsto (\mathcal{E}_{x_i,l_i}, l_i') \qquad \mathcal{E}_{x_i,l_i}(U) = \{s \mid s(x_i) \in l_i\}$$

$$l_i \subset \mathcal{E}_{x_i}$$

$$0 \to \underset{\deg d-1}{\mathcal{E}_{x_i,l_i}} \to \mathcal{E} \to \underset{\deg 1}{(\mathcal{E}_x/l_i)_{x_i}} \to 0$$

$$s \mapsto s_x \overset{\text{proj.}}{\mapsto} \mathcal{E}_x/l_i$$

$$s \mapsto 0 \Leftrightarrow s_x \in l_i$$

$$\mathcal{E}_{x_i,l_i} \hookrightarrow \mathcal{E}$$

$$(\mathcal{E}_{x_i,l_i})_{x_i} \to \mathcal{E}_x \to 0$$

$$\searrow l_i \nearrow$$

$l_i' := \ker$ Ex. $L_i^2(\mathcal{E}, x_i) \simeq (\mathcal{E}, x_i) \otimes \mathcal{O}(-1)$

construct a morphism $\mathrm{Bun}^1_{2,-1}(\mathbb{P}^1, x_i) \to \mathbb{P}^1_{\mathbb{C}}$

$(\mathcal{E}, l_i) \qquad \mathcal{E} \simeq \mathcal{O}(-1) \oplus \mathcal{O}$

$$L_1 L_2 L_3 L_4 (\mathcal{E}, l_i) = (\mathcal{E}', l_i')$$

$$\mathcal{E}' \subset \mathcal{E}$$

$\deg \mathcal{E}' = -5$ $\qquad$ $\mathcal{E}' = \mathcal{O}(-2) \oplus \mathcal{O}(-3)$

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook] & \mathcal{E} \\ \mathcal{O}(-2) \arrow[r, hook] & \mathcal{E}' \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

embedding

Now $0 \to \mathcal{O} \oplus \mathcal{O}(-2) \to \mathcal{E} \to \mathcal{O}_x \to 0$

deg $-2$ $\qquad$ deg $-1$ $\qquad$ deg $1$

skyscraper

$$\mathrm{Bun}^1_{2,-1}(\mathbb{P}^1, x_i) \Rightarrow \mathbb{P}^1_{\mathbb{C}}$$

$(\mathcal{E}, l_i) \mapsto x$[illegible]

$\mathbb{C}[\mathrm{Conn}_r(X)] = \mathbb{C}$

$\mathcal{O}_{\mathrm{Conn}_r}(X)$

$\updownarrow$

$\delta \in D\text{-mod}(\mathrm{Bun}_r(X))$

$G \supset B \supset V$

$\mathbb{C}^*[G/V]$

Section 4.4. [AF]

$\mathrm{Bun}'_{2,1} = (L, \eta) \qquad L \xrightarrow[\text{rank } 2]{} \mathbb{P}^1$

degree 1

$\eta = (\eta_i \subset L_{x_i},\ i = 1, 2, 3, 4)$

$\mathrm{End}(L, \eta) = \mathbb{C}$

$L \simeq \mathcal{O} \oplus \mathcal{O}(1) \qquad (\mathcal{O}(-1) \oplus \mathcal{O}(2)$ cannot occur )

$\exists!$ up to scaling $\mathcal{O}(1) \hookrightarrow L$

Rigidification

$P = \{(L, \eta, s) : (L, \eta) \in \mathrm{Bun}'_{2,1},\ s : \mathcal{O}(1) \hookrightarrow L\}$

$\mathrm{Aut}(L,\eta) = \mathbb{C}^* = \mathbb{G}_{m,\mathbb{C}}$ nontrivial

$\mathrm{Bun}'_{2,1}$ is a stack $\mathbb{C}^*$-gerbe

$$\begin{array}{c} P \longrightarrow \mathrm{Bun}'_{2,1} \\ \circlearrowleft \\ \mathbb{G}_m \end{array}$$

"coarse moduli space"

$\lambda(L,\eta,s) = (L,\eta,\lambda s)$

$P$ $\mathbb{G}_m$-torsor over $\mathrm{Bun}'_{2,1}$

$\mathrm{Bun}'_{2,1} = P \times B\mathbb{G}_m$

$\circlearrowleft$
$\mathbb{G}_m$

$\mathrm{Aut}(L,\eta,s) = \star$

Proof.

$$\begin{array}{ccc} & & L \\ & \overset{s}{\nearrow} & \downarrow \varphi \\ \mathcal{O}(1) & \underset{s}{\searrow} & L \end{array} \qquad \varphi(\eta) = \eta$$

$\mathrm{Aut}(L,\eta) = \mathbb{C}^* \Rightarrow \varphi = \lambda$

$\Rightarrow \lambda = 1$ $\square$

Coordinates $L_\eta := L_{\eta_1,\dots,\eta_4}$

$L_\eta(U) = \{ s \in L(U) : \forall i \; s(x_i) \in \eta_i \}$ $x_i \in U$

$\circledast$ $U \subset \mathbb{P}^1 - \{x_1,\dots,x_4\}$ $L_\eta|_U = L|_U$

$\mathrm{Aut}(L_\eta, \eta') \simeq \mathrm{Aut}(L, \eta) = \mathbb{C}^*$

$\deg L_\eta = 1 - 4 = -3$

$L_\eta \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2)$

$\mathcal{O}(-1) \underset{s'}{\hookrightarrow} L_\eta$ $\exists!$ up to scaling

$$
\begin{array}{ll}
2 & \tilde{P} = \{(L, \eta, s, s') : (L,\eta) \in \mathrm{Bun}'_{2,1}, \; s : \mathcal{O}(1) \hookrightarrow L \;\; s' : \mathcal{O}(-1) \hookrightarrow L_\eta\} \\
 & \quad \downarrow \mathbb{G}_m \\
1 & P \\
 & \quad \downarrow \mathbb{G}_m \\
0 & \mathrm{Bun}'_{2,1}
\end{array}
$$

$(L, \eta, s, s') \in \tilde{P}$

$\mathcal{O}(1) \hookrightarrow L$

$\mathcal{O}(-1) \hookrightarrow L_\eta$

$\mathcal{O}(-1) \oplus \mathcal{O}(1) \hookrightarrow L$

Ex. $L \simeq \mathcal{O} \oplus \mathcal{O}(1)$

$\mathcal{O}(1) \overset{i_1}{\hookrightarrow} \mathcal{O} \oplus \mathcal{O}(1)$

$\mathcal{O}(-1) \overset{i_2}{\hookrightarrow} \mathcal{O} \oplus \mathcal{O}(1)$

Hint. local $\rightsquigarrow$ about modules $\mathcal{O}_{X,x}$ (DVR)

$i_1 \oplus i_2$ is injective iff $i_2(\mathcal{O}(-1)) \not\subset i_1(\mathcal{O}(1))$

$\otimes \quad 0 \to \mathcal{O}(-1) \oplus \mathcal{O}(1) \hookrightarrow L \twoheadrightarrow F \to 0$

| | | | |
|---|---|---|---|
| rank | 2 | 2 | 0 |
| deg | 0 | 1 | 1 |

$F \simeq \mathcal{O}_q \qquad q \in \mathbb{P}^1$

$$\mathcal{O}(-1)_q \oplus \mathcal{O}(1)_q \xrightarrow{\varphi} L_q \twoheadrightarrow V \to 0 \qquad \dim V = 1$$

$\otimes$ not left exact

$q = \infty \qquad \mathcal{O}(-1)_q \oplus \mathcal{O}(1)_q \cong \mathbb{C} \oplus \mathbb{C}$

$$\ker\varphi \subset \mathcal{O}(-1)_q \oplus \mathcal{O}(1)_q$$

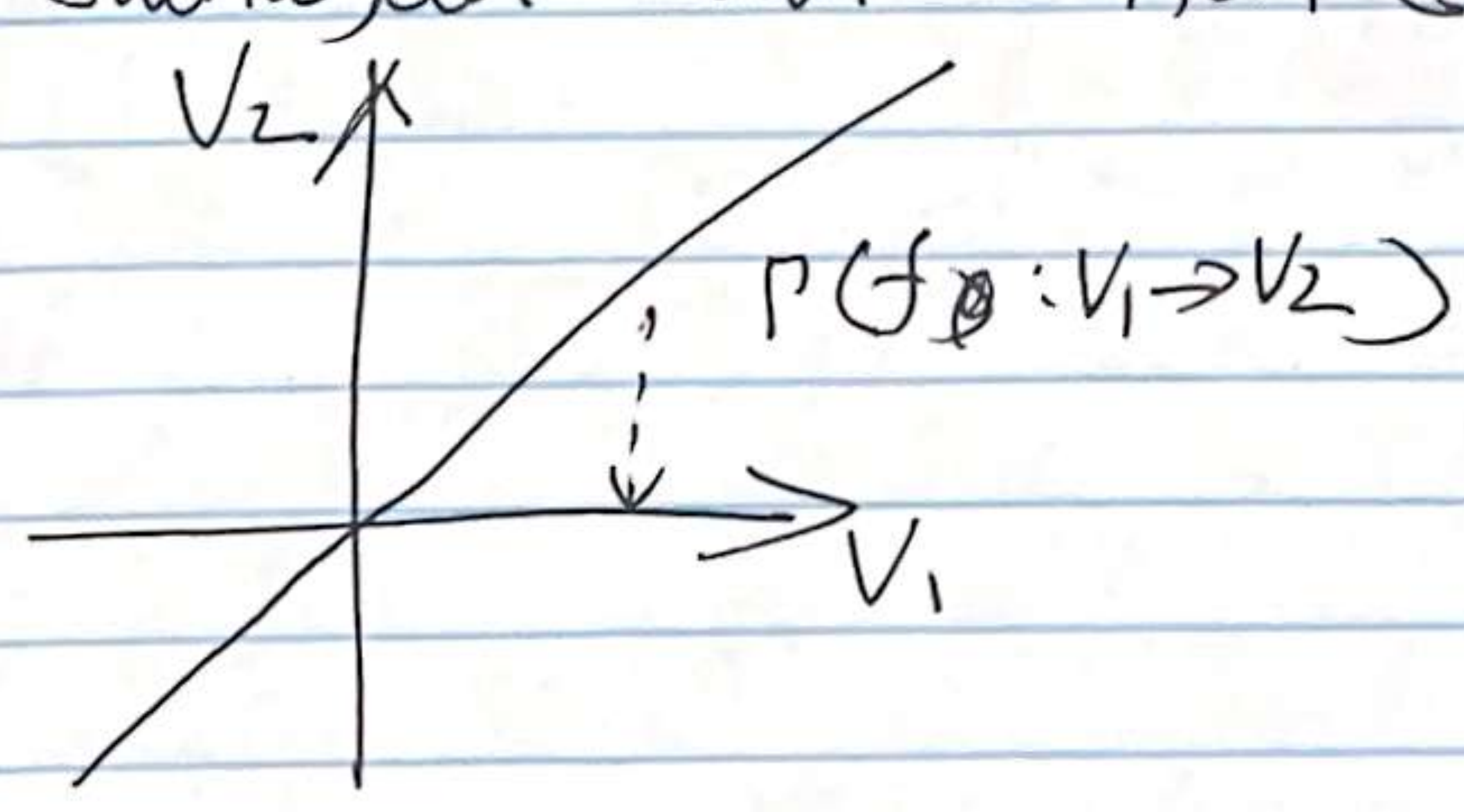
$$\in \mathbb{P}(\mathcal{O}(-1)_q \oplus \mathcal{O}(1)_q)$$

Ex. $\operatorname{Ker}\varphi \neq \mathcal{O}(1)_q$ Hint. otherwise $\varphi$ not inj.

Lemma. Let $\dim V_1 = \dim V_2 = 1$

1-dim subspaces in $V_1 \oplus V_2$ ($\neq V_2$)

identified with $\operatorname{Hom}(V_1, V_2)$



$$\operatorname{Ker}\varphi \iff p \in \operatorname{Hom}(\mathcal{O}(-1)_q, \mathcal{O}(1)_q) = \mathcal{O}(1)_q$$

$$\tilde{p}^{\circ} \longrightarrow \operatorname{Tot}(\mathcal{O}(2)) = T\mathbb{P}^1$$

$$(L, \eta, s, s') \longmapsto (q, p)$$

$$\downarrow \qquad \uparrow \; \mathbb{P}^1$$

Outline.

$$\tilde{P} \to T\mathbb{P}^1$$

$$\cdots \longmapsto (q, p')$$

$$\tilde{P} \hookrightarrow \operatorname{Tot}(\mathcal{O}(2) \oplus \mathcal{O}(2)) = T\mathbb{P}^1 \times_{\mathbb{P}^1} T\mathbb{P}^1$$

$$\cdots \longmapsto (q, p, p')$$

hypersurface $\hookrightarrow$ dim 3

$$pp' = f(q) \Leftarrow L_q \subset L$$

$$p, p' \in \mathcal{O}(2)_q \qquad pp' \in \mathcal{O}(4)_q$$

$$f = \prod_i (z - x_i)$$

$$\uparrow$$

$$H^0(\mathbb{P}^1, \mathcal{O}(4))$$

$$\mathcal{O}(-1) \hookrightarrow L_\eta$$

$$L_{\eta\eta'}(u) = \{ s \in L_{\eta^\circ}(u), \ \forall i \ \ s(x_i) \in \eta' \}$$

$$= L(-x_1 - x_2 - x_3 - x_4)$$

$\deg L_{\eta\eta'} = -7$

$L_{\eta\eta'} \simeq \mathcal{O}(-3)\oplus\mathcal{O}(-4)$

$$\begin{tikzcd} \mathcal{O}(-3) \arrow[r, hook, "s(-4)"'] & L_{\eta\eta'} \arrow[r, hook] & L_\eta \\ & \mathcal{O}(-1) \arrow[ur, hook, "s'"] & \end{tikzcd}$$

$$\mathcal{O}(-3)\oplus\mathcal{O}(-1) \xrightarrow{s(-4)\oplus s'} L_\eta \to \mathcal{O}_q \to 0$$

$$\mathcal{O}(-3)_q\oplus\mathcal{O}(-1)_q \xrightarrow{\varphi'} (L_\eta)_q$$

$$\ker \varphi' \leftrightarrow p'$$

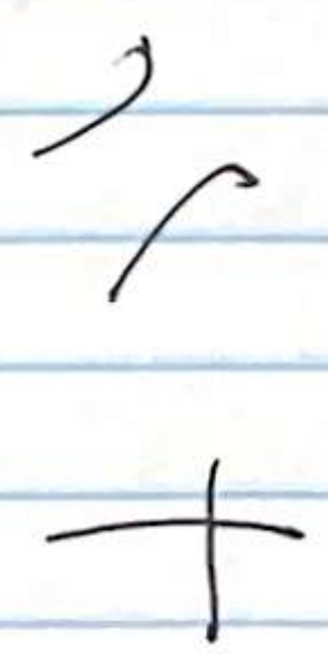
$\boxed{\text{WHY } p, p', q \text{ determine the point of } \tilde{P}}$

- $(p, q)$ determine $L$ as upper mod of $\mathcal{O}(-1)\oplus\mathcal{O}(1)$
- $(p', q)$ determine $L_\eta$
- $L_\eta$, $L$ determine $\eta$

$\tilde{P} \subset P' = \{(q,p,p'), q \in \mathbb{P}^2, p, p' \in \mathcal{O}(2), pp' = f(q)\}$

$$\begin{tikzcd} P' \arrow[d, "\pi"] \\ \mathbb{P}^1 \end{tikzcd} \qquad q$$

$\pi^{-1}(q) \simeq$



$q \neq x_i$ hyperbola

$q = x_i$

$q \neq x_i$ $f(q) \neq 0$ $qq' = \text{const.}$

$q = x_i$ $f(q) = 0$

$$\begin{array}{cc} G_m \times & \boxed{\begin{array}{c} G_m \\ \circlearrowright \\ S' \end{array}} \\ \circlearrowright & \\ S & \end{array}$$

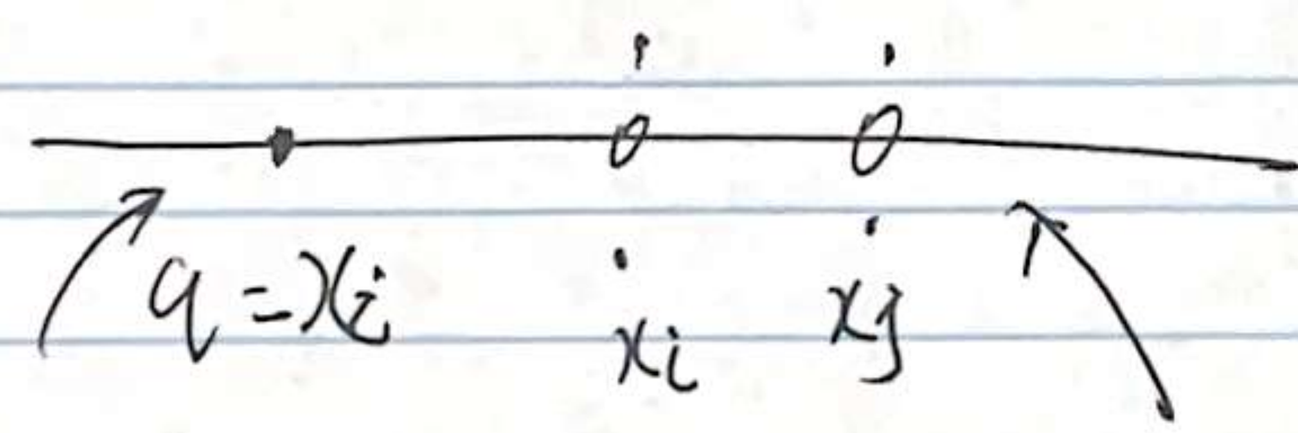
$a \in G_m \qquad a \cdot (q, p, p') = (q, ap, \frac{p'}{a})$

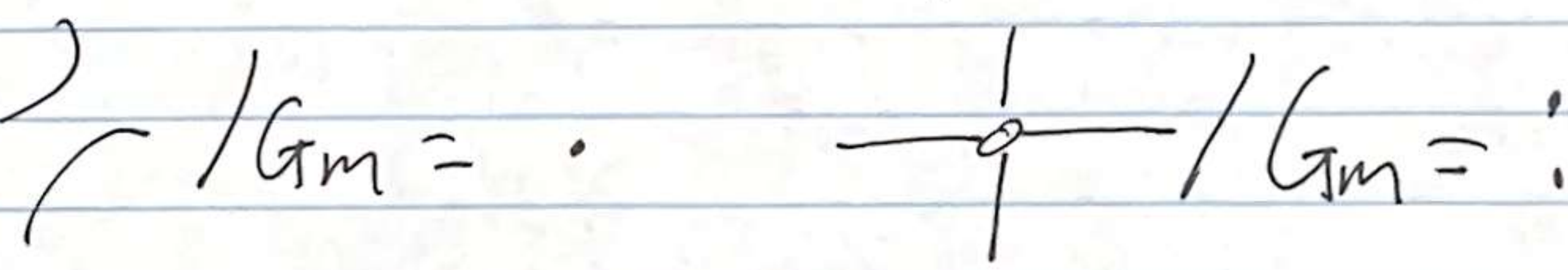
$\cdot$ $(0,0)$ trivial action

$\tilde{P} = P'$ – centers of the crosses

$\tilde{P} \xrightarrow{G_m} P$

$P = \tilde{P} / G_m$




$$\begin{array}{ccc} P' & \rightarrow & Bun_{2,1} \\ \cup & & \cup \\ \tilde{P} & \rightarrow & Bun'_{2,1} \end{array}$$

$$\begin{array}{ccc} (L, \eta, s, s') & \in & P' \\ \downarrow & & \downarrow \pi \\ q & & \mathbb{P}^1 \end{array}$$

$\pi$ is algebraic because constructions work in families

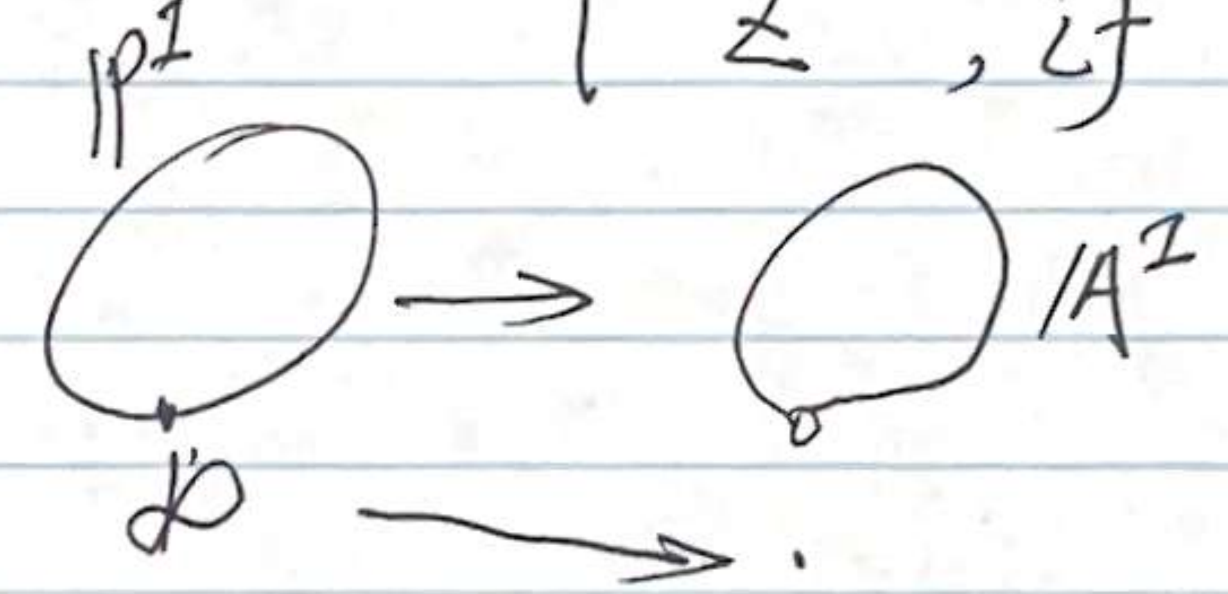
$\mathrm{Bun}_{2,d}(X) \longrightarrow \mathrm{Bun}_{2,d+2\deg \ell}(X)$ $\qquad E \mapsto E \otimes \ell$

$\ell \to X$ line bundle

only defined for $\mathbb{C}$-pts

WARNING. $\quad \mathbb{P}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}} \sqcup \mathrm{Spec}\,\mathbb{C}$

$$z \mapsto \begin{cases} \mathrm{Spec}\,\mathbb{C}, & \text{if } z = \infty \\ z, & \text{if } z \neq \infty \end{cases}$$




not even continuous

To define a map $\mathcal{X} \to \mathcal{Y}$, one needs to define a map $\mathcal{X}(S) \xrightarrow{f_S} \mathcal{Y}(S)$ for all schemes $S$

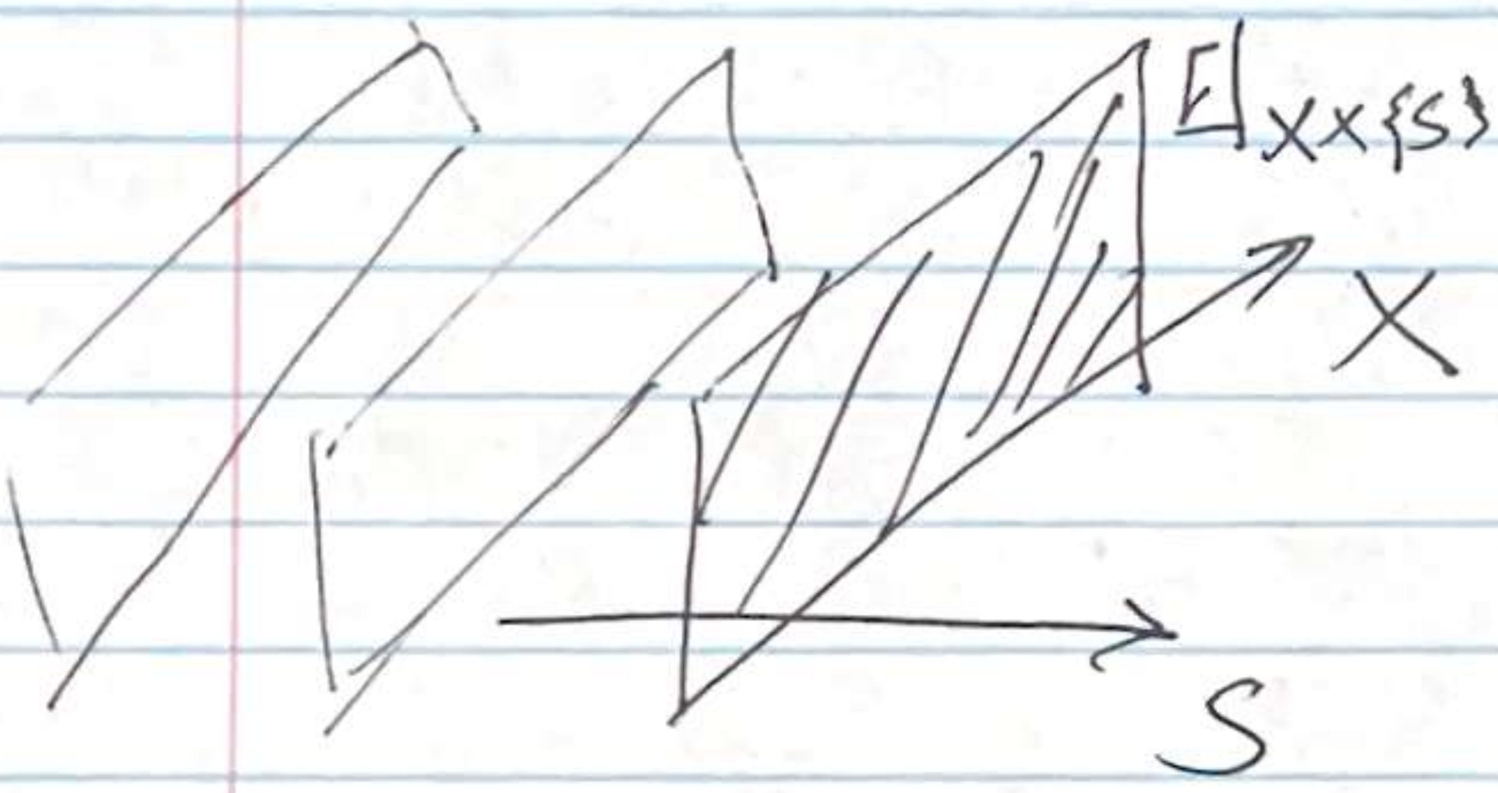
$\mathcal{X}(S) = \mathrm{Mor}(S, \mathcal{X})$

$\varphi : S \to S'$

$$\begin{array}{ccc} \mathcal{X}(S) & \xrightarrow{f_S} & \mathcal{Y}(S') \\ \varphi^* \downarrow & & \downarrow \varphi^* \\ \mathcal{X}(S') & \to & \mathcal{X}(S') \end{array}$$

$(\mathrm{Bun}_{2,d}(X))(S) = \mathrm{Mor}(S, \mathrm{Bun}_{2,d}(X))$

$$\begin{array}{c} \| \\ \{E \to X\times S,\ \text{rank } 2,\ \forall s\in S\ \deg E|_{X\times\{s\}} = d\} \end{array}$$




$$\begin{array}{ccc} \mathrm{Bun}_2(X)(S) & \longrightarrow & \mathrm{Bun}_2(X)(S) \\ \rotatebox{90}{$\in$} & & \\ E & \longmapsto & E \otimes p_1^*\ell \\ \downarrow & & \\ X\times S & & \\ \downarrow p_1 & & \\ X & & \end{array}$$

do the actions in families

Ex. $\ell \in \mathbb{P}^1_{\mathbb{C}} \longmapsto (\mathcal{O}\oplus\mathcal{O})^{(u)}_{\infty,\ell} := \{ s \in \mathcal{O}\oplus\mathcal{O}(u) \mid s(\infty) \subset \ell \}$

$\ell \subset \mathbb{C}^2$, $\mathbb{C}^2 = (\mathcal{O}\oplus\mathcal{O})(\infty)$

$$\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathrm{Bun}_2(\mathbb{P}^1)\ ?$$

$$\mathbb{P}^1_{\mathbb{C}} = \{S \to \mathbb{P}^1\} = \{\ell \subset \mathcal{O}_S \oplus \mathcal{O}_S\}$$

line subbundle

$$\{\ell \subset \mathcal{O}_S \oplus \mathcal{O}_S\} \longmapsto \{\mathcal{E}(u) \subset \mathcal{O}_{\mathbb{P}^1\times S} \oplus \mathcal{O}_{\mathbb{P}^1\times S}\}$$

$$\begin{array}{c} \downarrow \\ \mathbb{P}^1_t \times S \end{array}$$

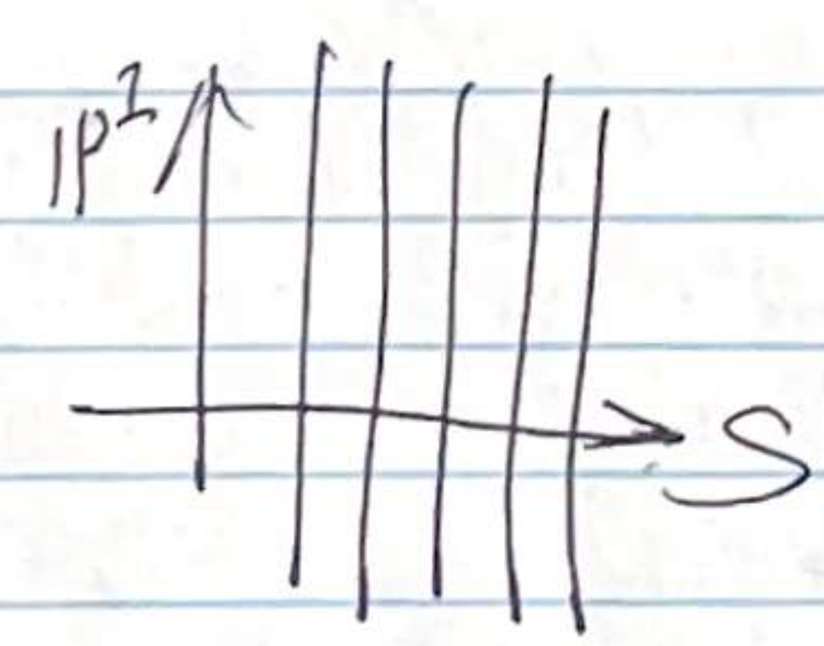
$$\mathcal{E}(u) = \{s : \begin{array}{c} s|_{\infty\times S} \\ \cap \\ \mathcal{O}_S \oplus \mathcal{O}_S \end{array} \subset \ell\}$$

Lemma. Let $\mathcal{E} \to \mathbb{P}^1 \times S$ coherent sheaf, then

$\mathcal{E}$ is a v.b. of rank 2 $\Leftrightarrow$ ~~$\mathcal{E}|_{\mathbb{P}^1\times\{s\}}$ is a v.b. $\forall s \in S$.~~

$\forall u \in \mathbb{P}^1 \times S$

$\dim \mathcal{E}_u = 2$.




surface

$\ell \to X$ $\quad$ $\mathrm{Tot}(\ell) \to X$

l.b. curve

locally $\approx U \times \mathbb{C}$

$\begin{array}{c} U \\ \cap \\ X \end{array}$

$\mathrm{Tot}(\ell) \to \mathrm{Bun}_2(X)$

$(x, p \in \ell_x)$

$\to \mathbb{A}(p,1) \subset (\mathcal{L}\oplus\mathcal{O})_x$

lower mod at $x$ along $(p,1)$

$$\mathcal{E} = (\mathcal{L}+\mathcal{O})_{x,(p,1)}$$

$$(\mathrm{Tot}(\mathcal{L}))(S) = \{ S \xrightarrow{\varphi} x,\ \sigma \in H^0(S, \varphi^*\mathcal{L}) \}$$

$$\mathcal{E} \subset \mathcal{L}\oplus\mathcal{O}$$

$$\{ (s_1, s_2) \in \mathcal{L}\oplus\mathcal{O} : \varphi^* s_1 = (\varphi^* s_2)\sigma \}$$

$$\cancel{H^0(S, \varphi^*\mathcal{L}\oplus\varphi^*\mathcal{O})}$$

$$\varphi^* s_1 \in H^0(S, \varphi^*\mathcal{L})$$

$$\varphi^* s_2 \in H^0(S, \mathcal{O}_S)$$

$$(s_1(x), s_2(x)) \in \mathbb{A}(p,1) \iff s_1(x) = s_2(x)p$$

Last time. $\mathrm{Tot}(\mathcal{O}(2)\oplus\mathcal{O}(2)) \rightrightarrows \mathrm{Tot}(\mathcal{O}(2))$

$p' = \{pp' = f(q)\} \subset$

$$\begin{array}{ccc} (q,p,p') & \longmapsto & (q,p) \\ \swarrow & & \downarrow \\ (q,p') & & \mathrm{Bun}_2(\mathbb{P}^1) \end{array}$$

$$\mathcal{E} \subset \mathcal{O}(2)\oplus\mathcal{O}$$

$$\mathcal{E}' \subset \mathcal{O}(2)\oplus\mathcal{O}$$

$$\mathcal{E}'\otimes\mathcal{O}(-4) \subset \mathcal{O}(-2)\oplus\mathcal{O}(-4)$$

$$\| \\ \mathcal{E}''$$

$$\begin{array}{ccccc} (q,p,p') & \mapsto & \mathcal{O}(2)\oplus\mathcal{O} & \supset & \mathcal{O}(-2)\oplus\mathcal{O}(-4) \\ & & \cup & & \cup \\ & & \mathcal{E} & \supset ? & \mathcal{E}'' \end{array}$$

Lemma. $(q,p,p') \in \mathbb{P}^1 \iff \mathcal{E}'' \subset \mathcal{E}$ and

~~supp~~ $\mathcal{E}/\mathcal{E}'' \simeq \mathcal{O}_{x_1}\oplus\cdots\oplus\mathcal{O}_{x_4}$

$\mathrm{Par}_2(\mathbb{P}^1, x_1, x_2, x_3, x_4)$

$l_i := \mathrm{image}(\mathcal{E}''_{x_i} \to \mathcal{E}_{x_i}) \subset \mathcal{E}_{x_i}$

Prop. $\mathrm{Par}_{2,1}(\mathbb{P}^1, x_1, \cdots, x_4) \cong \{ \mathcal{E}'' \subset \mathcal{E} \quad \mathrm{rk}\,\mathcal{E}=2, \deg\mathcal{E}=1$

$\mathcal{E}/\mathcal{E}'' \simeq \mathcal{O}_{x_1}\oplus\cdots\oplus\mathcal{O}_{x_4} \}$

$\{\mathcal{E}, l_1, \cdots, l_4\} \mapsto \mathcal{E}'' = \mathcal{E}_{x_1,l_1,\cdots,x_4,l_4}$

$\mathrm{Tot}(\mathcal{O}(2)\oplus\mathcal{O}(2)) \supset p'$

$$\downarrow$$

$$\left(\begin{array}{ccc} \mathcal{O}(2)\oplus\mathcal{O} & \supset & \mathcal{O}(-2)\oplus\mathcal{O}(-4) \\ \cup & & \cup \\ \mathcal{E} & & \mathcal{E}'' \end{array}\right) \supset (\mathcal{E}/\mathcal{E}'' \simeq \mathcal{O}_{x_1}\oplus\cdots\oplus\mathcal{O}_{x_4})$$

$\cancel{\mathrm{Par}_{2,1}(\mathbb{P}^1, x_i)}$ open $= \mathrm{Par}_{2,1}(\mathbb{P}^1, x_i)$

$\cancel{\{\mathrm{End}(\mathcal{E}, l_i) = \mathbb{C}\}}$

$$\begin{array}{ccc} \mathcal{H} & \to & \mathrm{Bun} \\ \pi \downarrow & & \\ \mathrm{Bun} & \supset & \mathcal{O}\oplus\mathcal{O}(2) \end{array}$$

$\pi^{-1}(\mathcal{O}\oplus\mathcal{O}(2)) = \mathrm{Tot}(\mathcal{O}(2))$

semicontinuity

Lemma. $E \downarrow X\times S$ $\qquad s\in S \mapsto \dim \mathrm{End}(E_s)$

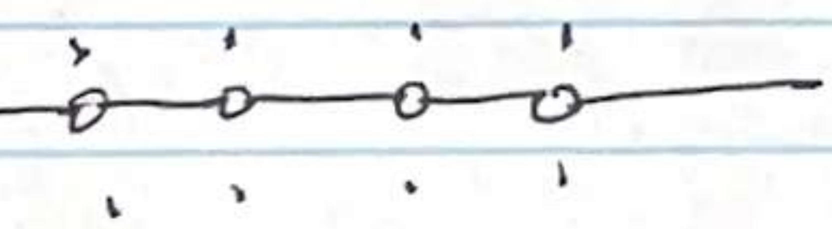
$P' \supset P''$ $\qquad X \supset c$

$P' \subset$

$\mathrm{Par}_{2,1}(\mathbb{P}^1, x_i) \underset{\text{open}}{\supset} \mathrm{Par}'_{2,1}(\mathbb{P}^1, x_i)$ by semicontinuity

$\mathrm{Par}'_{2,1}(\mathbb{P}^1, x_i) = \{\mathrm{End}(\mathcal{E}, l_i) = \mathbb{C}\}$

$\mathrm{Par}' \cong B\mathbb{G}_m \times P$

$P$ 

$\mathcal{M} = \{(E,\nabla) : \nabla : \mathcal{E} \to \mathcal{E}\otimes\Omega(x_1+\cdots+x_4)\}$

$\downarrow$ affine bundle

$\mathrm{Par}'$

$\mathcal{M}$ locally $\mathrm{Par}' \times \mathbb{A}^1_{\mathbb{C}}$

$\mathcal{M} = M\times B\mathbb{G}_m$ $\quad M$ is a smooth surface

$M$ separated $\hookrightarrow \mathbb{P}^N_{\mathbb{C}}$

$\downarrow \mathbb{C}$

$P$ not separated

$\mathrm{Par}_1 = \{(E,\eta)\}$ $\quad E \xrightarrow{2} \mathbb{P}^1 \supset D = x_1 + x_2 + x_3 + x_4$

$\cup$ $\quad \eta = \{\eta_i\} \quad \eta_i \subset E_{x_i} \quad \deg E = 1$

$\mathrm{Par}_1' = \{\mathrm{Aut}(E,\eta) = \mathbb{C}^*\}$

$\mathrm{Conn} = \{(E,D) : D : E \to E \otimes \Omega^1_X(D)\}$

+ conditions at the poles

near $x_i$ $\quad D \sim \frac{A}{z} dz + d$

+ regular

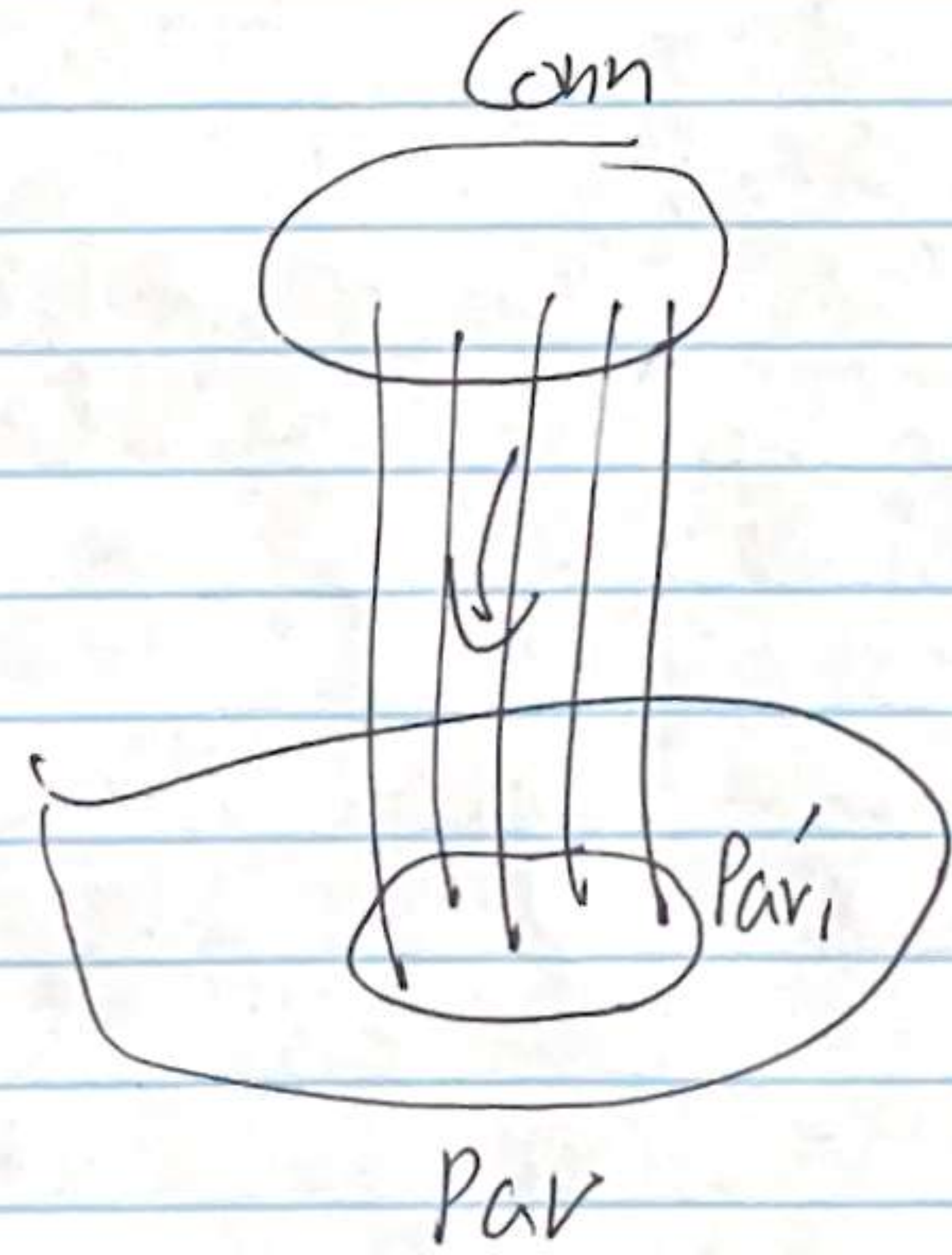
$\mathrm{Conn} \xrightarrow{\pi} \mathrm{Par}$

$A$ has eigen. $\lambda_i^{\pm}$

$\lambda_i^{\pm}$ generic $\quad \Sigma(\lambda_i^+ + \lambda_i^-) = -1$

Thm. (i) $\pi(\mathrm{Conn}) = \mathrm{Par}_1'$

(ii) Non-empty fibers of $\pi \simeq \mathbb{C}$




$X$ Curve proj. smooth $/\mathbb{C}$

$$\begin{array}{ccc} \mathrm{Conn}_X & = & \{(E,D)\} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_X & = & E \end{array}$$

fiber over $E$ is either empty or

$$\approx H^0(X, End(E) \otimes \Omega_X^1)$$

Fix $E^r \to X$

sections $\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_r \end{pmatrix}$

(1) $X = \bigcup_\alpha X_\alpha$

$E|_{X_\alpha} \approx X_\alpha \times \mathbb{C}^r$

$\nabla_\alpha$ conn on $E|_{X_\alpha}$

$S_{\alpha\beta}| + S_{\beta\gamma} + S_{\gamma\alpha}| = 0$

On $X_\alpha \cap X_\beta$ $S_{\alpha\beta} := \nabla_\alpha|_{X_\alpha \cap X_\beta} - \nabla_\beta|_{X_\alpha \cap X_\beta}$

$\in H^0(X_\alpha \cap X_\beta, End(E) \otimes \Omega_X^1)$

$E \rightsquigarrow E \otimes \Omega_X^1$ connection is not $\mathcal{O}_X$-linear

$\nabla_1(fS) = S \otimes df + f\nabla_1 S$

$\nabla_2(fS) = S \otimes df + f\nabla_2 S$

$(\nabla_1 - \nabla_2)(fS) = f(\nabla_1 - \nabla_2)S$

$\Rightarrow \nabla_1 - \nabla_2 \in Hom(E, E \otimes \Omega_X)$

$\cong$

$Hom(\mathcal{O}_X, E^\vee \otimes E \otimes \Omega_X)$

$\|$

$H^0(X, End(E) \otimes \Omega_X^1)$

$$[S_{\alpha\beta}] \in H^1(X, End(E) \otimes \Omega_X^1)$$

Lemma. (i) $[S_{\alpha\beta}]$ depends only on $E$

Atiyah class $a(E)$

(ii) $E$ has a connection $\Leftrightarrow a(E) = 0$

Pf. Assume $[S_{\alpha\beta}] = 0$

$$S_{\alpha\beta} = \varphi_\alpha| - \varphi_\beta| \qquad \varphi_\alpha \in H^0(X_\alpha, End(E) \otimes \Omega_X^1)$$

$$\nabla_\alpha' = \nabla_\alpha - \varphi_\alpha \qquad \text{Conn on } E|_{X_\alpha}$$

$$\nabla_\alpha'|_{X_\alpha \cap X_\beta} - \nabla_\beta'|_{X_\alpha \cap X_\beta}$$

$$= \nabla_\alpha|_{X_\alpha \cap X_\beta} - \nabla_\beta|_{X_\alpha \cap X_\beta} - (\varphi_\alpha|_{X_\alpha \cap X_\beta} - \varphi_\beta|_{X_\alpha \cap X_\beta})$$

$$= S_{\alpha\beta} - S_{\alpha\beta}$$

$= 0$ $\qquad \exists$ connection ✓

$\square$

Ex. $rk\, E = 1$

$End(E) = E^\vee \otimes E = \mathcal{O}_E$

rank 1 + section

$$H^1(X, \mathcal{O}_X \otimes \Omega_X^1) = H^1(X, \Omega_X) \simeq \mathbb{C}$$

Serre's duality on a proj. curve

$$H^1(X, \underset{\substack{\uparrow \\ \text{v.b.}}}{\mathcal{E}}) \text{ dual to } H^0(X, \mathcal{E}^\vee \otimes \Omega_X)$$

In particular

$$H^1(X, \Omega_X) \text{ dual to } H^0(X, \Omega_X^\vee \otimes \Omega_X)$$
$$= H^0(X, \mathcal{O}_X) \cong \mathbb{C}$$

$\ell$. l.b.

$$a(\ell) \in H^1(X, \Omega_X) = \mathbb{C}$$

$$\ell|_{X_\alpha} \overset{\sigma_\alpha}{\approx} X_\alpha \times \mathbb{C}$$

$$\ell|_{X_\alpha \cap X_\beta} \overset{\sigma_\alpha|}{\underset{\sigma_\beta|}{\rightrightarrows}} (X_\alpha \cap X_\beta) \times \mathbb{C}$$

$$\mu_{\alpha\beta} = \frac{\sigma_{\alpha\beta}}{\sigma_\alpha} : X_\alpha \cap X_\beta \to \mathbb{C}^\times$$

deg $\ell$

$$\left[\frac{d\mu_{\alpha\beta}}{\mu_{\alpha\beta}}\right] \in H^1(X, \Omega_X)$$

$$\cancel{a(\ell) = \sum \operatorname{res}\left(\frac{d\mu_{\alpha\beta}}{\mu_{\alpha\beta}}\right) = \deg \ell}$$

Ex. $\mathcal{O}(n) \to \mathbb{P}^1$

$$\mathcal{O}(n)|_{\mathbb{A}^1} \simeq \mathbb{A}^1 \times \mathbb{C} \qquad \mathcal{O}(n)|_{\mathbb{P}^1 - \{\infty\}} \simeq (\mathbb{P}^1 - \infty) \times \mathbb{C}$$

$$\mu_{\alpha\beta} = z^n \qquad \mathbb{A}^* \to \mathbb{C}^*$$

$$\frac{d\mu_{\alpha\beta}}{\mu_{\alpha\beta}} = \frac{n z^{n-1} dz}{z^n} = \frac{n}{z} dz$$

$$\Omega^1(\mathbb{A}^2 - 0)$$
$$\psi$$
$$f dz \qquad f \in \mathbb{C}[z^{-1}, z]$$

$$a_{-n} z^{-n} dz + \cdots + a_{-2} z^{-2} dz + a_{-1} z^{-1} dz + a_0 dz + \cdots$$

$$\Omega^1(\mathbb{A}^1)$$

$$f(w) dw \qquad w = z^{-1}$$

$$\Rightarrow dw = -z^{-2} dw$$

[Griffiths & Harris]

$\ell$ has a connection $\Leftrightarrow$ $\deg \ell = 0$

$E$ has a connection $\nabla$

$\Lambda^{top} E$ has $\Lambda^{top} \nabla \Rightarrow \deg \Lambda^{top} E = 0$

$$\deg \Lambda^{top} E = \deg E$$

Converse? NO

$E = E' \oplus E'' \Rightarrow E'$ has a connection

$E' \hookrightarrow E \xrightarrow{\nabla} E \otimes \Omega_X^1$

$\deg E' = 0$

Thm. [Atiyah] $E$ has a connection $\Leftrightarrow$

every direct summand has degree $0$

"Pf" $E = E_1 \oplus \cdots \oplus E_n$ $E_i$ indecom-

enough to construct $\nabla_i : E_i \to E_i \otimes \Omega_X^1$

enough to show indecom $E$ of $\deg 0$ has

a connection

$$a(E) \in H^1(X, End(E) \otimes \Omega^1)$$
$$\cong H^0(X, (E^\vee \otimes E \otimes \Omega^1)^\vee \otimes \Omega^1)^\vee$$
$$= H^0(X, End(E))^\vee$$
$$= End(E)^\vee$$

Fact. $E$ indecomposable

$\Leftrightarrow \{\forall \varphi \in End(E),\ \varphi = \text{scalar} + \text{nilp}\}$

$\forall \varphi$ $\langle a(E), \varphi \rangle = 0$

$\langle a(E), Id_E \rangle = \deg E = 0$ $\Rightarrow a(E) = 0$

$\varphi^n = 0$ $\langle a(E), \varphi \rangle = 0$ $\square$

$(X, p)$ $\qquad$ $\operatorname{rk} E = 2$

$(E, \nabla)$ $\quad$ $\nabla : E \to E \otimes \Omega_X(p)$

near $p$ $\quad$ $\nabla = d + \frac{A}{z} dz + \cdots$ regular

eigenvalues of $A$ are $\lambda^{\pm}$

Lemma. $\exists !$ line $\ell \subset E_p$ such that if $s \in E$

s.t. $s(p) \in \ell$

$\nabla(s) = \frac{\lambda^{+} s}{z} + \cdots$ regular

$A \circlearrowright E_p$ $\quad$ $A \in \operatorname{End}(E_p)$

Lemma. $A$ does not depend on the local trivialization (and on coordinates.)

$\operatorname{res} \nabla := A \in \operatorname{End}(E_p)$

$\ell := \lambda^{+}$- eigenspace of $A$

$(E, \nabla) \mapsto (E, \ell) = \tilde{E}$

$(E, \nabla_1)$ and $(E, \nabla_2)$ with same $\tilde{E}$ ?

$\nabla_1 - \nabla_2 : E \to E \otimes \Omega_X(p)$

$\mathcal{O}_X$- linear

$s \in E \quad s(p) \in l$

$$(\nabla_1 - \nabla_2)(s) \in E \otimes \Omega_X \subset E \otimes \Omega_X(p)$$

$s(E_p) \subseteq l$

$$\operatorname{End}^0(\tilde{E}) = \{ s \in \operatorname{End}(E) : s(l) = 0 \}$$

$$\cap$$

$$\operatorname{End}(\tilde{E}) = \{ s \in \operatorname{End}(E) : s_p(l) \subset l \}$$

$$\begin{pmatrix} \lambda & * \\ 0 & \mu \end{pmatrix} \qquad \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$$\nabla_1 - \nabla_2 \in H^0(X, \operatorname{End}^0(E) \otimes \Omega_X(p))$$

$$a(\tilde{E}) \in H^1(X, \operatorname{End}^0(E) \otimes \Omega_X(p))$$

Prop. is Serre dual to $\operatorname{End}(\tilde{E})$

$$\operatorname{End}(\tilde{E}) = H^0(X, \operatorname{End}(E))$$

$$\operatorname{End}^0(\mathcal{E}) \otimes \operatorname{End}(E) \to \mathcal{O}_X$$

$$\langle a(\tilde{E}), \varphi \rangle \qquad \varphi \in \operatorname{End}(\tilde{E}) = \mathbb{C}^\times$$

$$\langle a(\tilde{E}), \mathrm{Id}_E \rangle = \deg E + \underbrace{\lambda^- + \lambda^+}_{} = 0$$

why we have $\sum \lambda_i^\pm = -1$

$\mathbb{P}^1, \ x_1 + \cdots + x_4$

$\operatorname{End}(E, \eta)$ pairs $H^1(\mathbb{P}^1, \operatorname{End}^0(E) \otimes \Omega(D))$

$\exists\, \varphi \in End(E, \eta)$ not scalar

$\Rightarrow \langle \varphi, a(E,\eta) \rangle \neq 0 \iff$ genericity

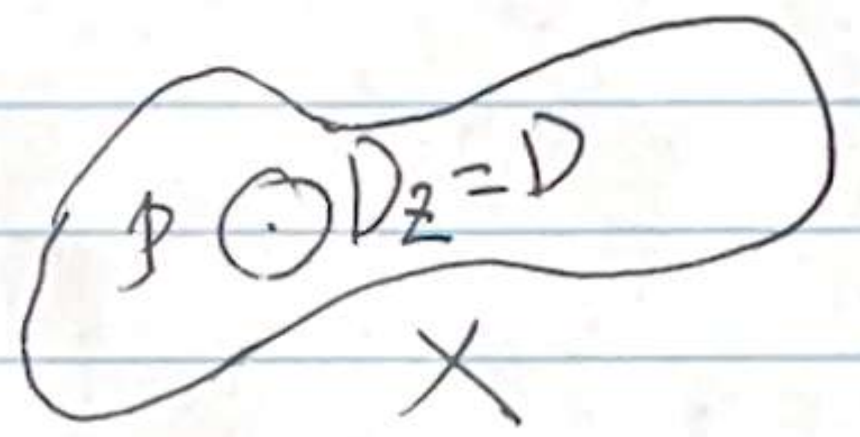
assumptions on $\lambda^{\pm}$

$\langle \varphi_0, a(E,\eta) \rangle = 0 \Rightarrow$ linear equation on $\lambda^{\pm}$

$X$ – smooth proj. curve $/\mathbb{C}$

$\mathcal{E}$ – coherent sheaf (e.g. v.b.)

$H^1(X, \mathcal{E}) =$




Principal parts of $\mathcal{E}$ at $p$: $\mathcal{E}(\dot{D})/\mathcal{E}(D)$

$\dot{D} = D - p$

$$\mathcal{E}(X-p)|_{D-p} \to \mathcal{E}(\dot{D})/\mathcal{E}(D)$$

Thm. $H^1(X, \mathcal{E}) \simeq \mathcal{E}(\dot{D})/\mathcal{E}(D) + \mathcal{E}(X-p)$

Pf. $0 \to \mathcal{E} \to \mathcal{E}(\infty p) \to \mathcal{E}(\dot{D})/\mathcal{E}(D) \to 0$

Čech cohomology

e.g. $\mathcal{E} = \Omega_X$ $\quad \mathcal{E}(\dot{D}) = \{ (a_{-n} z^{-n} + a_{1-n} z^{1-n} + \cdots + a_1 z^{-1} + a_0 + \cdots ) dz \}$

$$\Omega_X(\dot{D})/\Omega_X(D) = \{(a_{-n} z^{-n} + \cdots + a_1 z^{-1}) dz\}$$

$$H^1(X, \Omega) = (\Omega_X(\dot{D})/\Omega_X(D))/\Omega_X(X-p)$$

$$H^1(X,\Omega) \cong \mathbb{C}$$

$a^{-1}$

$p_1, \cdots, p_n \in X$

$p_i \in D_i$

$$\left(\bigoplus \mathcal{E}(\dot{D}_i)/\mathcal{E}(\bar{D}_i)\right)/H^0(X - p_1 - \cdots - p_n, \mathcal{E}) \simeq H^1(X, \mathcal{E})$$

$\mathcal{L} \to X$ line bundle Atiyah class

$\exists p \in X$ $\mathcal{L}|_{X-p}$ is trivial (assumed)

$$\mathcal{O}_{X-p} \xrightarrow[\sim]{\alpha} \mathcal{L}|_{X-p} \quad \text{trivialization}$$

$$\mathcal{O}_D \xrightarrow[\sim]{\beta} \mathcal{L}|_D$$

$$\begin{array}{ccc} 1 \mapsto s & & \\ \downarrow & & \\ f & & \end{array}
\qquad
\begin{tikzcd} \mathcal{O}_{D-p} \arrow[r, "\alpha|_{D-p}"] \arrow[d, "\cdot f"'] & \mathcal{L}|_{D-p} \arrow[dl, "\beta^{-1}|_{D-p}"] \\ \mathcal{O}_{D-p} & \end{tikzcd}
\qquad f \in \mathcal{O}_X(D-p)$$

$\nabla_1 s = 0$ $\nabla_1$ on $\mathcal{L}|_{X-p}$

$$\nabla_2(fs) = 0$$
$$\|$$
$$f\nabla_2(s) + df \otimes s$$

Now $\nabla_1|_{D-p} - \nabla_2|_{D-p} = \frac{df}{f} \rightsquigarrow H^1(X, \Omega_X)$

$a(\mathcal{L}) = \mathrm{res}_p \frac{df}{f} = \mathrm{ord}_p f$ where $f = z^n f_0$
$\qquad\qquad\qquad\qquad\qquad\quad \| \qquad\qquad\quad f_0(p) \neq 0$
$\qquad\qquad\qquad\qquad\qquad\quad n$

$$a(\mathcal{L}) = \sum_i \mathrm{res}_{p_i} \frac{df_i}{f_i} = \sum_i \mathrm{ord}_{p_i} f_i = \deg \mathcal{L}$$

Residues $\quad \alpha \in \Omega_X(D-p) \quad \mathrm{res}_p \alpha \in \mathbb{C}$

$$\alpha = \cdots + \frac{a_1}{z} dz$$

(i) does not depend on the coordinate

(ii) linear $\quad \mathrm{res}_p(f\alpha) = f(p)\,\mathrm{res}_p \alpha$

$\mathcal{E} \to X$

$$\mathcal{E} \otimes \Omega_X(D-p) \xrightarrow{\mathrm{res}} \mathcal{E}_p$$

$$\mathcal{E}(D-p) \times \Omega_X(D-p) \to \mathcal{E}_p$$

$$(e, \alpha) \mapsto \mathrm{res}_p \alpha \cdot e(p)$$

$f \in \mathcal{O}_X(D-p) \quad (fe, \alpha) = \mathrm{res}_p(\alpha) \cdot f(p) e(p)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$ by linearity

$\qquad\qquad (e, f\alpha) = \mathrm{res}_p(f\alpha) \cdot e(p)$

$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X(p) \qquad$ coherent sheaves have connections

Want to define: $\mathrm{res}_p \nabla \in \mathrm{End}(\mathcal{E}_p) \qquad \updownarrow$

$e \in \mathcal{E}_p \xrightarrow{\text{lift}} \tilde{e} \in \mathcal{E}(D-p) \quad \tilde{e}(p) = e \qquad$ v.b.s

$$\nabla \tilde{e} \in (\mathcal{E} \otimes \Omega_X(p))(D) \subset (\mathcal{E} \otimes \Omega_X)(D-p)$$

$$\mathrm{res}_p \nabla(e) := \mathrm{res}_p (\nabla \tilde{e}) \in \mathcal{E}_p$$

another $\hat{e}(p) = e \quad (\hat{e} - \tilde{e})(p) = 0$

$$\hat{e} - \tilde{e} = z \cdot e' \qquad e' \in \mathcal{E}(D - p)$$

$$\nabla(\hat{e} - \tilde{e}) = \nabla(z e') = \underbrace{z \nabla e'}_{\text{no poles}} + \underbrace{e' \otimes dz}_{\text{no poles}} \in \mathcal{E} \otimes \Omega_X(p)$$

$$\mathrm{res}_p \nabla(\hat{e} - \tilde{e}) = 0$$

Ex. Trivialize $\mathcal{E}$ ove $D$

identify $\mathcal{E}_p \simeq \mathbb{C}^r$ $\qquad r = \text{rank } \mathcal{E}$

$$\nabla = d + \frac{A}{z} dz + \text{regular}$$

$$\mathrm{res}\, \nabla = A \in \mathrm{Mat}_{r \times r} = \mathrm{End}(\mathcal{E}_p)$$

$$\left\{ \begin{array}{l} (\mathcal{E}, \nabla) : \mathcal{E} \xrightarrow{r} X \quad \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_X(p) \\ \text{eigenvalues of } \mathrm{res}_p \nabla \text{ are } \lambda_1, \dots, \lambda_r \end{array} \right\}$$

$$\mathrm{Conn}(X, p, \lambda_1, \dots, \lambda_r) \qquad \lambda_i \neq \lambda_j \text{ if } i \neq j$$

Lemma. If $(\mathcal{E}, \nabla) \in \mathrm{Conn}$,

then $\deg \mathcal{E} = -\lambda_1 - \dots - \lambda_r$.

$$\mathrm{res} \Lambda^r \nabla = \mathrm{tr} \nabla$$

Rem. can be ~~no~~ empty since $\lambda_i \in \mathbb{C}$ may not $\in \mathbb{Z}$

$$\mathrm{Conn}(X, p, \lambda_1, \cdots, \lambda_r) \xrightarrow{\Pi} \mathrm{Par}_r(X, p) \ni \tilde{\mathcal{E}}$$

$$= \{ \mathcal{E} \xrightarrow{r} X, \ \text{full flag} \quad 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = \mathcal{E}_p \}$$

$F_i$ - $i$-dim subspace of $\mathcal{E}_p$

$$\downarrow \sim Fl_r$$

$$\mathrm{Bun}_r$$

$\mathrm{res}_p \nabla \in \mathrm{End}\, \mathcal{E}_p$

$\mathcal{E}_p = l_1 \oplus \cdots \oplus l_r$

$\mathrm{res}_p \nabla$ act on $l_i$ by $\lambda_i$

$F_i := l_1 \oplus \cdots \oplus l_i$

$\tilde{\mathcal{E}} = (\mathcal{E}, 0 = F_0 \subset \cdots \subset F_{r-1} \subset F_r = \mathcal{E}_p)$

$U \subset X$ open

$$(\mathrm{End}(\tilde{\mathcal{E}}))(U) = \begin{cases} \{ \mathcal{E}|_U \to \mathcal{E}|_U \} \text{ if } p \notin U \\ \left\{ \begin{array}{l} \mathcal{E}|_U \xrightarrow[\varphi]{} \mathcal{E}|_U \\ \varphi_p : \mathcal{E}_p \to \mathcal{E}_p \text{ s.t.} \\ \varphi_p(F_i) \subset F_i \end{array} \right\} \text{ if } p \in U \end{cases}$$

$\mathrm{End}^0(\tilde{\mathcal{E}})(U) = \{ \varphi_p(F_i) \subset F_{i-1} \text{ if } p \in U \}$

$\mathrm{End}^0(\tilde{\mathcal{E}}) \subset \mathrm{End}(\tilde{\mathcal{E}}) \subset \mathrm{End}(\mathcal{E})$

In eigenbasis of $res_p \nabla$

$$\varphi_p = \begin{pmatrix} * & * & & \\ 0 & * & & * \\ \vdots & 0 & \ddots & \\ \vdots & \vdots & & \ddots \\ 0 & 0 & 0 & * \end{pmatrix}$$ for $End(\tilde{\mathcal{E}})$

$$\varphi_p = \begin{pmatrix} 0 & * & * \\ 0 & \ddots & \\ \vdots & & \ddots \\ \vdots & & & 0 \\ 0 & \cdots & \cdots & \end{pmatrix}$$ for $End^0(\tilde{\mathcal{E}})$

Theorem. Assume $\Pi(\mathcal{E}_0, \nabla_0) = \tilde{\mathcal{E}} = (\mathcal{E}_0, F_i)$

Then $\underset{\substack{\uparrow \\ \text{can be empty}}}{\Pi^{-1}(\tilde{\mathcal{E}})} \simeq \underbrace{H^0(X, End^0(\tilde{\mathcal{E}}) \otimes \Omega_X(p))}_{\text{vector space}}$

$\Pi(\mathcal{E}, \nabla) = \tilde{\mathcal{E}} \qquad \mathcal{E} = \mathcal{E}_0$

$\nabla_0, \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega(p)$

$\Phi := \nabla_0 - \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega(p) \quad \mathcal{O}_X\text{-linear}$

$\Phi(fs) = \nabla(fs) - \nabla_0(fs)$

$= (f\nabla s + s \otimes df) - (f\nabla_0 s + s \otimes df)$

$= f(\Phi)(s)$

$\Phi \in Hom_{\mathcal{O}}(\mathcal{E}, \mathcal{E} \otimes \Omega(p)) = Hom_{\mathcal{O}}(\mathcal{O}_X, (\mathcal{E}^\vee \otimes \mathcal{E}) \otimes \Omega(p))$

$= \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_X, \mathrm{End}(\mathcal{E}) \otimes \Omega(p))$

$= H^0(X, \mathrm{End}(\mathcal{E}) \otimes \Omega(p))$

trivialize $\mathcal{E}$ in a nhd $D$ of a point

$s_1, \cdots, s_r \in \mathcal{E}(D)$ $\qquad$ $s_i(p)$ basis of $\mathcal{E}_p$

$F_i = \langle s_1(p), \cdots, s_i(p) \rangle$ = span of $\lambda_1, \cdots \lambda_i$

$\nabla_0 = d + \frac{A_0}{z} dz + \text{reg.}$ $\qquad$ eigenspaces of $A_0$

Claim. $A_0 = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$ $\qquad$ $A_0 = \mathrm{res}_p \nabla_0$

Lin. Alg. let $A_0 \in \mathrm{Mat}_{r \times r}$ with eigenvalues

$\lambda_1, \cdots, \lambda_r$ $(\lambda_i \neq \lambda_j)$

Assume $e_1, \cdots, e_r$ eigenbasis and

$\langle e_1, \cdots e_i \rangle$ is "standard" $F_i$

$\Rightarrow A_0 = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$

e.g. $r = 2$ $\quad$ $\langle (e_1) \rangle = \langle (1, 0) \rangle$

$\downarrow$

$\lambda_1$-eigenvector $\qquad \Rightarrow \begin{pmatrix} \lambda_1 & * \\ 0 & * \end{pmatrix}$

$\uparrow$

$\lambda_2$

$$\nabla = d + \frac{A}{z}dz + \text{reg.} \qquad A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$$

$$\Phi = \nabla - \nabla_0 = \frac{A - A_0}{z}dz + \text{reg.}$$

$$A - A_0 = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

$$\Phi \in \mathrm{End}^0(\tilde{\mathcal{E}}) \iff \Phi_p = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

$$\iff \Phi = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} + z \cdot B$$

Now $\left( \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} + z \cdot B \right) \frac{dz}{z}$

$$= \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \frac{dz}{z} + \underbrace{B\,dz}_{\text{reg.}}$$

OTOH $\quad \Phi \in H^0(X, \mathrm{End}^0(\tilde{\mathcal{E}}) \otimes \Omega_X(p))$

$$\cap$$

$$H^0(X, \mathrm{End}(\mathcal{E}) \otimes \Omega_X(p)) = \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{E} \otimes \Omega_X(p))$$

$$\nabla = \nabla_0 + \Phi : \mathcal{E} \to \mathcal{E} \otimes \Omega_X(p)$$

$$\Pi(\mathcal{E}_0, \nabla) = (\mathcal{E}_0, F_i)$$

$$\mathrm{Conn}(X, p, \lambda_1, \dots, \lambda_r) \to \mathrm{Bun}_r(X, p)$$

$$\text{V.B.} \longleftrightarrow GL_n$$

$$\mathcal{E} \rightsquigarrow \varphi$$

$$\text{End}(\mathcal{E}) = \text{ad}\,\varphi$$