Introduction to Stacks

Stacks

varieties (or schemes) $\rightsquigarrow$ points

stacks $\rightsquigarrow$ points with group attached to them

Why? Moduli problems and

quotient problems

Motivation I

$X = \mathbb{P}^1_{\mathbb{C}}$

$\{ E \to \mathbb{P}^1_{\mathbb{C}}, \text{ rank } 2 \text{ v.b.s} \}$

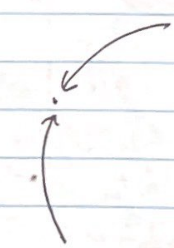
$E \simeq \mathcal{O}(n) \oplus \mathcal{O}(m), \; n, m \in \mathbb{Z}$

$Bun_2(\mathbb{P}^1) = \{ n, m \in \mathbb{Z}, n \leq m \}$

Ex. $\exists \; E \to \mathbb{P}^1 \times \mathbb{A}^1$

$E_t := E|_{\mathbb{P}^1 \times t} \simeq \mathcal{O} \oplus \mathcal{O} \quad t \neq 0$

$E_0 \simeq \mathcal{O}(1) \oplus \mathcal{O}(-1)$



$\Rightarrow \quad \mathcal{O}(1) \oplus \mathcal{O}(-1) \in \overline{\mathcal{O} \oplus \mathcal{O}}$

$Ext^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}(1)^\vee \otimes \mathcal{O}(-1))$

$= H^1(\mathbb{P}^1, \mathcal{O}(-2)) \simeq \mathbb{C}$

$$\mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{p_1} \mathbb{P}^1$$

$$0 \to p_1^*\mathcal{O}(-1) \to E \to p_1^*\mathcal{O}(1) \to 0$$

on $\mathbb{P}^1 \times \mathbb{A}^1 = \mathbb{P}^1 \times \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$

$t \neq 0$ $\quad 0 \to \mathcal{O}(-1) \to E \to \mathcal{O}(1) \to 0$

Exercise: construct E by gluing two trivial v.b.

on $\mathbb{P}^1 \times \mathbb{A}^1 = ((\mathbb{P}^1 - 0) \times \mathbb{A}^1) \cup (\mathbb{A}^1 \times \mathbb{A}^1)$

$$\mathrm{Aut}(\mathcal{O}(-n) \oplus \mathcal{O}(n)) \underset{\text{open}}{\subset} \mathrm{End}(\mathcal{O}(-n) \oplus \mathcal{O}(n))$$

$$\begin{aligned}
&= \mathrm{Hom}(\mathcal{O}(-n) \oplus \mathcal{O}(n), \mathcal{O}(-n) \oplus \mathcal{O}(n)) \\
&= \mathrm{Hom}(\mathcal{O}(-n), \mathcal{O}(-n)) \oplus \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n)) \oplus \cdots \\
&= H^0(\mathbb{P}^1, \mathcal{O}) \oplus H^0(\mathbb{P}^1, \mathcal{O}) \oplus H^0(\mathbb{P}^1, \mathcal{O}(2n)) \\
&\quad \oplus H^0(\mathbb{P}^1, \mathcal{O}(-2n)) \\
&= \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^{2n+1} \oplus \begin{Bmatrix} 0, & n > 0 \\ \mathbb{C}, & n = 0 \end{Bmatrix} \\
&= \begin{cases} \mathbb{C}^{2n+3}, & n > 0 \\ \mathbb{C}^4, & n = 0 \end{cases}
\end{aligned}$$

$$\dim \mathrm{Aut}(\mathcal{O}(-n) \oplus \mathcal{O}(n)) = \begin{cases} \cancel{\mathbb{C}}^4, & n = 0 \\ \cancel{\mathbb{C}^{2n+3}}\ 2n+3, & n > 0 \end{cases}$$

$\mathcal{O} \oplus \mathcal{O}$ $\qquad$ $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ $\qquad$ $\mathcal{O}(-2) \oplus \mathcal{O}(2)$

$\mathrm{Aut}(\mathcal{O} \oplus \mathcal{O})$ $\qquad$ $\mathrm{Aut}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$

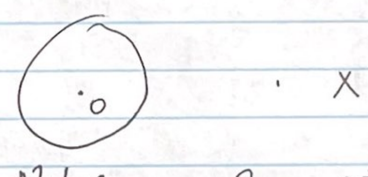
-4 $\qquad$ -5 $\qquad$ -7

closure contains $\mathcal{O}(-1) \oplus \mathcal{O}(1)$

explained by $-4 > -5$

Motivation II

$\mathbb{G}_{m_{\mathbb{C}}} = \mathbb{C}^*$



$\mathbb{G}_m \curvearrowright \mathbb{A}^1$ $\qquad$ $\mathbb{A}^1 / \mathbb{G}_m = \mathrm{Spec}\, \mathbb{C}[s]^{\mathbb{G}_m} = \mathrm{Spec}\, \mathbb{C}$

$t \cdot a = ta$ $\qquad$ $f(s) \mapsto f(ts)$

$t \cdot bs^n = bt^n s^n$

"$\mathbb{A}^1 / \mathbb{G}_m$" set theoretically

$\cdot \leftarrow x$

Stacky quotient $\qquad$ two points

$\{0\}$	$\{\mathbb{C}^\times\}$
$\mathbb{G}_m$	1
-1	0

Motivation III $\qquad$ Functor of pts

$X \in Sch_{\mathbb{C}}$

$Sch_{\mathbb{C}}^{op} \to Set$

$S \mapsto X(S) := Mor_{\mathbb{C}}(S, X)$

$Sch_{\mathbb{C}} \hookrightarrow Funct(Sch_{\mathbb{C}}^{op}, Sets) =: PSh(Sch_{\mathbb{C}}^{op})$

fully faithfuly by Yoneda's lemma

$X(\mathbb{C})$ = closed pts of X (X of finite type)

$X(Spec\, \mathbb{C}[\varepsilon]/\varepsilon^2)$ = tangent vectors to X

$Aff_{\mathbb{C}} \subset Sch_{\mathbb{C}}$

$\cong$

$\mathbb{C}\text{-}alg^{op}$

$Sch_{\mathbb{C}} \hookrightarrow Funct(Sch_{\mathbb{C}}^{op}, Sets) \to Funct(\mathbb{C}\text{-}alg, Sets)$

Hilbert Scheme

X proj scheme

$Hilb(X)$ classfies all closed subschemes of X

$(Hilb(X))(S) = \left\{ \begin{array}{l} Z \subset X \times S \\ \text{flat over } S \end{array} \right\}$

$S \to Hilb(X)$

connected components are projective

$Bun_2(\mathbb{P}^1)$ $\quad S \to Bun_2(\mathbb{P}^1)$

$(Bun_2(\mathbb{P}^1))(S) = \{$ families of v.b. parameterized by $\mathbb{P}^1 \}$

$s \in S$ $\quad E|_{\mathbb{P}^1 \times \{s\}}$ $= \left\{ \begin{array}{c} E \\ \downarrow \\ \mathbb{P}^1 \times S \end{array} \text{ v.b. of rank } 2 \right\}$

$F \in Bun_2(S)$ $\quad E := p^*_{\mathbb{P}^1 \times S \to S} F \to \mathbb{P}^1 \times S$

non-trivial

Exercise : $\forall s \in S$ $\quad E_s$ is trivial

Now $\{\mathcal{O} \oplus \mathcal{O}\} \subset Bun_2(\mathbb{P}^1)$

$E \rightsquigarrow S \xrightarrow{f} Bun_2(\mathbb{P}^1)$

$\forall s \in S$ $\quad f(s) = \mathcal{O} \oplus \mathcal{O}$

$\Rightarrow f$ is constant $\quad S \to \{\mathcal{O} \oplus \mathcal{O}\}$

$\Rightarrow$ $Bun_2(\mathbb{P}^1)$ does not exist as a scheme

Def. Lax-functor is a "functor"

$\mathbb{C}$-alg $\to$ Groupoids

$\mathrm{Bun}_2(\mathbb{P}^1)(S) = \left\{ \begin{array}{l} E \Rightarrow \mathbb{P}^1 \times S, \text{ morphisms are} \\ \text{isomorphisms of v.b.} \end{array} \right\}$

$\mathrm{Bun}_2(\mathbb{P}^1)(\mathrm{Spec}\,\mathbb{C}) = \{ E \Rightarrow \mathbb{P}^1_{\mathbb{C}}, \text{ isomorphisms} \}$

S is a set

$\{ G_s \text{ is a group} \mid s \in S \}$

Cat $\underline{G}$

$$\mathrm{Ob}(\underline{G}) = S$$

$$\mathrm{Mor}(s, s') = \begin{cases} \phi, & s \neq s' \\ G_s, & s = s' \end{cases}$$

$f \in \mathrm{Mor}(s, s')$ $g \in \mathrm{Mor}(s', s'')$

$s = s' = s'' \Rightarrow f \circ g = \text{mult. in } G_s$

$\underline{G}$ is a groupoid

Lemma. Suppose $\mathcal{C}$ is a small groupoid

$\Rightarrow \exists\, S.\ \{G_s \mid s \in S\}$ $\mathcal{C} \underset{\text{equiv.}}{\simeq} \underline{G}$

Pf. $S = \mathcal{C}/\mathrm{iso} \ni s$ $s = \text{isoclass of } x \in \mathcal{C}$

$G_s := \mathrm{Aut}(x)$

$\boxtimes$

$\mathrm{Ob}\,\mathcal{C} = \{ \mathcal{O}(n) \oplus \mathcal{O}(m) \mid n \leq m \}$

$G_{\mathcal{O}(n) \oplus \mathcal{O}(m)} = \mathrm{Aut}(\mathcal{O}(n) \oplus \mathcal{O}(m))$

Schemes : $\mathrm{Aff}^{op}_{/R} \to \mathrm{Sets}$

Stacks : $\mathrm{Aff}^{op}_{/R} \to \mathrm{Grpds}$

Groupoid : $\{ G_s \mid s \in S \}$

$$\mathrm{ob}(G) = S$$

$$\mathrm{Mor}(s, s') = \begin{cases} \phi, & s \neq s' \\ G_s, & s = s' \end{cases}$$

X - scheme

$$(\mathrm{Bun}_n(X))(S) = \left\{ \begin{array}{l} \text{ob: } \begin{array}{c} E \\ \downarrow \\ X \times S \end{array}, \text{ rank } n \text{ v.b.} / X \times S \\ \text{mor} \left(\begin{array}{c} E_1 \quad E_2 \\ \searrow \ \swarrow \\ X \times S \end{array} \right) = \text{isomorphisms} \end{array} \right\}$$

$\cancel{\mathrm{Bun}_n(*)}$

$$(\mathrm{Bun}_n(\mathrm{Spec}\, \mathbb{C}))(S) = \left\{ \begin{array}{l} E \to S \quad + \\ \text{isomorphisms} \end{array} \right\}$$

$$(\mathrm{Bun}_n(\mathrm{Spec}\, \mathbb{C}))(\mathrm{Spec}\, \mathbb{C}) = \left\{ \begin{array}{l} * \\ \mathrm{Mor}(*, *) = GL(n, \mathbb{C}) \end{array} \right\}$$

F - lax functor

$$F : \mathrm{Aff}/\mathbb{C} \longrightarrow \mathrm{Grpds}$$

$$S \longmapsto F(S)$$

$f : S' \to S''$ $\quad f^* = F(f) : F(S') \to F(S)$

functor

① $F(\mathrm{Id}_S) = \mathrm{Id}_F$

② $F(f \circ g) \underset{lh_{f,g}}{\simeq} F(g) \circ F(f)$

③
$$\begin{tikzcd}
F(f \circ g \circ h) \arrow[r, "lh_{(f\circ g),h}"] \arrow[d, "lh_{f,g\circ h}"'] & F(h) \circ F(f \circ g) \arrow[d, "F(h)\circ lh_{f,g}"] \\
F(g \circ h) \circ F(f) \arrow[r, "lh_{g,h} \circ F(f)"] & F(h) \circ F(g) \circ F(f)
\end{tikzcd}$$

commutes

Ex.

$S \to S'$ $\qquad X \times S \xrightarrow{id_X \times f} X \times S'$

$$\left\{ \begin{matrix} E \\ \downarrow \\ X \times S' \end{matrix} \right\} \xrightarrow{f^*} \left\{ \begin{matrix} E \\ \downarrow \\ X \times S \end{matrix} \right\}$$

$(g \circ f)^* \simeq f^* \circ g^*$

Stack on étale not enough

topology: $\begin{cases} fppf \\ \text{smooth} \end{cases}$

$X/_{\mathbb{C}}$ scheme $\qquad\qquad$ $\mathbb{C}$-space = fppf sheaf on $Aff/_{\mathbb{C}}^{op}$

$Aff/_{\mathbb{C}}^{op} \to Sets$

$S \longmapsto Mor(S, X)$

$$\begin{array}{l} S_\alpha \\ \pi_\alpha \big\downarrow \text{ flat maps, finitely presented} \\ S \end{array}$$

$\bigcup_\alpha \pi_\alpha(S_\alpha) = S$

$\varphi_\alpha \in Mor(S_\alpha, X)$

$\varphi_\alpha|_{S_\alpha \times_S S_\beta} = \varphi_\beta|_{S_\alpha \times_S S_\beta} \quad \forall \alpha, \beta$

$\Rightarrow \exists! \varphi \in Mor(S, X)$

$\varphi|_{S_\alpha} = \varphi_\alpha$

Lax Functor $\quad F: Aff/\mathbb{C}^{op} \to Grpds$ is called a $\mathbb{C}$-stack

(i) $\forall x, y \in F(U)$, $U \in Aff/\mathbb{C}$

define $\quad Isom(x, y): Aff/_U^{op} \to Sets \qquad U' \xrightarrow{f} U$

$Isom(x, y)(U') = Isom(f^*x, f^*y)$

$f^*: F(U) \to F(U')$

Isom like "Hom"

$U'' \xrightarrow{f'} U' \xrightarrow{f} U$

$(f')^* : \mathrm{Isom}(f^*x, f^*y) \to \mathrm{Isom}((f')^* f^* x,$
$(f')^* f^* y)$

Ex. $\mathrm{Bun}_n(\mathrm{Spec}\,\mathbb{C})(\mathrm{Spec}\,\mathbb{C}) = \left\{ \begin{matrix} \star \\ GL(n,\mathbb{C}) \end{matrix} \right\}$

$\mathrm{Isom}(\star, \star) : \mathrm{Aff}/_{\mathbb{C}}^{op} \to \mathrm{Sets}$

$U' \in \mathrm{Aff}/_{\mathbb{C}}^{op} \quad U' \xrightarrow{f} \mathrm{Spec}\,\mathbb{C} = U$

$f^* \star = U \times \mathbb{C}^n \to U$

$\mathrm{Isom}(\star, \star)(U') = \mathrm{Isom}(\mathbb{C}^n \times U', \mathbb{C}^n \times U')$

$= \mathrm{Mor}(U', GL_n(n, \mathbb{C}))$

represented by scheme

$\boxed{1^{st}\text{ axiom}}$ of $\mathbb{C}$-stacks : $\forall U,\ x, y \in F(U)$

<u>Isom</u>(x,y) is an fppf sheaf on $\mathrm{Aff}/_U$

$F : \mathrm{Aff}/_{\mathbb{C}}^{op} \to \mathrm{Grpds} \quad U_\alpha \xrightarrow{\varphi_\alpha} U$ fppf

$x_\alpha \in F(U_\alpha) \qquad U_{\alpha\beta} := U_\alpha \times_U U_\beta = U_{\beta\alpha}$

$\downarrow \pi_{\alpha\beta}$

U_α

$\forall\, \alpha \cap \beta \qquad x_\alpha|_{U_{\alpha\beta}} := F(\pi_{\alpha\beta})\, x_\alpha$

$\qquad\qquad\qquad\qquad \in$

$\qquad\qquad\qquad\qquad F(U_{\alpha\beta})$

$\varphi_{\alpha\beta} : x_\alpha|_{U_{\alpha\beta}} \simeq x_\beta|_{U_{\alpha\beta}}$

Descent datum for F

$$U_\alpha \xrightarrow{\varphi_\alpha} U$$

$\forall \alpha : \; x_\alpha \in F(U_\alpha)$

$\forall \alpha, \beta : \; \varphi_{\alpha\beta} : x_\alpha|_{U_{\alpha\beta}} \simeq x_\beta|_{U_{\alpha\beta}}$

cocycle condition $\varphi_{\alpha\circ\beta} \circ \varphi_{\gamma\circ\alpha} = \varphi_{\gamma\circ\beta}$ in

$\mathrm{Iso}(x_\gamma|_{U_{\alpha\beta\gamma}},$

$x_\beta|_{U_{\alpha\beta\gamma}})$

$F(U_{\alpha\beta\gamma})$

$\boxed{2^{nd} \text{ axiom}}$ of $\mathcal{C}$-stacks

every descent datum is effective

$\exists\; x \in F(U)$

$\psi_\alpha \in \mathrm{Isom}(x|_{U_\alpha}, x_\alpha)$ (in $F(U_\alpha)$)

s.t. $\forall \alpha, \beta$: $\varphi_{\alpha\beta} \circ \psi_\alpha|_{U_{\alpha\beta}} = \psi_\beta|_{U_{\alpha\beta}}$ (in $F(U_{\alpha\beta})$)

$\mathrm{Bun}_n(\mathrm{Spec}\,\mathbb{C})$ is a stack

$\mathrm{Bun}_n(X)$ is a $\mathbb{C}$-stack

$G \circlearrowright X$

$X/G = ?$

If action is free, the "often" X/G exists

and $X \to X/G$ is a G-torsor

Always X/G exists as a stack

$X \to X/G$ is a G-torsor

in the category of stacks

$S \to \mathrm{Spec}\,\mathbb{C}$

$(X/G)(S) = \mathrm{Mor}(S, X/G)$

$$\begin{tikzcd} E \arrow[r, "G\text{-equiv}"] \arrow[d] & X \arrow[d, "G"] \\ S \arrow[r] & X/G \end{tikzcd} \qquad = \{ E \xrightarrow{G} S,\ E \to X,\ G \text{ equiv.} \}$$

$E := X \times_{X/G} S$

$\mathrm{Spec}\,\mathbb{C}/GL_n \simeq \mathrm{Bun}_n(\mathrm{Spec}\,\mathbb{C})$

$X(S) = \mathrm{Mor}(S, X)$

$\mathbb{C}$-space functor $\mathrm{Aff}^{op}/\mathbb{C} \to \mathrm{Sets}$

(+ gluing conditions)

$\mathbb{C}$-stack $\mathrm{Aff}^{op}/\mathbb{C} \xrightarrow{\mathcal{X}} \mathrm{Grpds}$

$S' \twoheadrightarrow S$, $x \in \mathcal{X}(S')$

fppf cover

1-morphism $\forall S \in \mathrm{Aff}/\mathbb{C}$ $S'' \xrightarrow{g} S' \xrightarrow{f} S$

$$\begin{tikzcd} \mathcal{X}(S) \arrow[r, "F(S)"] \arrow[d, "\mathcal{X}(f)"'] \arrow[dr, phantom, "\varphi_f"] & \mathcal{Y}(S) \arrow[d, "\mathcal{Y}(f)"] \\ \mathcal{X}(S') \arrow[r, "F(S')"'] & \mathcal{Y}(S') \end{tikzcd}$$

isomorphic

+ relation between $\varphi_{f\circ g}$, φ_f, φ_g

$\mathcal{X} \xrightarrow{F, G} \mathcal{Y}$

$F \simeq G$ isomorphic

$\forall S$ $F(S) \xrightarrow[\sim]{h_S} G(S)$

$\{\mathbb{C}\text{-spaces}\} \subset \{\mathbb{C}\text{-stack}\}$

$\{\mathrm{Sets}\} \subset \{\mathrm{Groupoids}\}$

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

fiber product

$$\begin{tikzcd} X \times_S S' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

base change

$X \mapsto X \times_S S'$

$Sch/S \to Sch/S'$ If $S' = \mathrm{Spec}\,\mathbb{C}$, it is the fiber of points

$(X \times_Y Z)(S) = X(S) \times_{Y(S)} Z(S)$ fiber product of sets + condition

In Sets

$$\begin{tikzcd} & A \arrow[d, "f"] \\ C \arrow[r, "g"] & B \end{tikzcd}$$

$A \times_B C = \{(a, c) \mid f(a) = g(c)\}$

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "G"] & \mathcal{B} \end{tikzcd}$$

$\mathcal{A} \times_{\mathcal{B}} \mathcal{C}$ fiber product of category

$$\mathrm{Ob}(\mathcal{A}\times_{\mathcal{B}}\mathcal{C}) = \{\overset{(x,y,\varphi)}{x\in \mathrm{Ob}(\mathcal{A}), y\in \mathrm{Ob}(\mathcal{C})}$$

$$F(x)\xrightarrow[\varphi]{\sim} G(y)\}$$

$\mathrm{Mor}((x,y,\varphi),(x',y',\varphi')) = \{(\alpha,\beta) : \alpha : x\to x',$

$\beta : y\to y'$

$$\begin{tikzcd} F(x) \arrow[r, "\varphi"] \arrow[d, "F(\alpha)"'] & G(y) \arrow[d, "G(\beta)"] \\ F(x') \arrow[r, "\varphi'"'] & G(y') \end{tikzcd}$$

Ex.

G - abstract group

$BG = \{\star, \mathrm{Aut}(\star) = G\}$

$\star \to BG$

$$\begin{tikzcd} \mathrm{Set} \to G \arrow[r] \arrow[d] & \star \arrow[d] \\ \star \arrow[r] & BG \end{tikzcd}$$

$\{(\star, \star, g) \mid g\in G\} = G$

discrete groupoid

$\mathcal{A}\times_{\mathcal{B}}\mathcal{C} \leftarrow \mathcal{A}, \mathcal{C}$ set

also a set

$\mathcal{X}, \mathcal{Y}, \mathcal{Z} : \mathrm{Aff}/_{\mathbb{C}}^{op} \to \mathrm{Grpds}$

$$\begin{tikzcd} & \mathcal{X} \arrow[d, "F"] \\ \mathcal{Z} \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

$\mathcal{X}(S)\times_{\mathcal{Z}(S)}\mathcal{Y}(S) =: (\mathcal{X}\times_{\mathcal{Y}}\mathcal{Z})(S)$

Fact: fibered product of $\mathbb{C}$-stacks is a $\mathbb{C}$-stack

Def. let $F: \mathcal{X} \to \mathcal{Y}$ be a 1-morphism

(1) schematic if $\forall \eta \quad S \to \mathcal{Y}$, where $S \in Aff/\mathbb{C}$

$S \times_{\mathcal{Y}} \mathcal{X}$ is represented by

$\mathbb{C}$-stack   a scheme

Rem. $S \times_{\mathcal{Y}} \mathcal{X}$ is a $\mathbb{C}$-space if

$\forall$ $F(S): \mathcal{X}(S) \to \mathcal{Y}(S)$ are faithful

(2) schematic $F: \mathcal{X} \to \mathcal{Y}$ is smooth if

$\forall S: \quad S \times_{\mathcal{Y}} \mathcal{X} \to S$ is smooth

Fibers

$$\begin{tikzcd} X_\eta \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r, "\eta"'] & Y \end{tikzcd}$$

Ex. $G \curvearrowright X$

$X \to X/G$ is smooth

$$(X/G)(S) = \left\{ \begin{tikzcd} E \arrow[r, "\text{equiv.}"] \arrow[d, "G"] & X \\ S \end{tikzcd} \right\}$$

$$X(S) \to (X/G)(S)$$

$$\begin{array}{c} \in \\ f \end{array} \mapsto \left\{ \begin{tikzcd} S \times G \arrow[r, "f \times id"] \arrow[d] & X \times G \arrow[r, "act"] & X \\ S \end{tikzcd} \right\}$$

$$\cancel{X \to X/G}$$

$$\begin{tikzcd} & S \arrow[d] \\ X \arrow[r] & X/G \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f"] \arrow[d] & X \\ S \end{tikzcd}$$

~~we~~ need to find

$$\begin{tikzcd} & S(T) \arrow[d] \\ X(T) \arrow[r] & X/G(T) \end{tikzcd} \qquad \begin{array}{l} \alpha : T \to S \\ \beta : T \to X \end{array}$$

$$\beta \mapsto \left\{ \begin{tikzcd} G \times T \arrow[r] \arrow[d] & G \times X \arrow[r] & X \\ T \end{tikzcd} \right\}$$

$$S(T) \to X/G(T)$$

$\varphi \Updownarrow$ ← for fiber product

they are isomorphic

$$\alpha \mapsto \left\{ \begin{tikzcd} \alpha^* E \arrow[r] \arrow[d] & E \arrow[r, "\alpha"] \arrow[d] & X \\ T \arrow[r] & S \end{tikzcd} \right\}$$

trivializations of α^*E = sections $T \to \alpha^*E$

= $\mathrm{Mor}_S(T, E)$

$X \times_{X/G} S = E \leftarrow$ scheme

smooth in scheme theory = smooth in fibers

Ex. closed embedding not

A stack is algebraic (= Artin)

(i) $\exists$ a scheme P and a smooth schematic surjective

$$P \twoheadrightarrow \mathcal{X}$$

(ii) $\forall S \in \mathrm{Aff}/R$ $\quad x, y \in \mathcal{X}(S)$

$\mathrm{Iso}(x, y)$ is represented by a scheme

Prop. Let G be an affine algebraic group acting on X, then X/G is an algebraic stack

sketch:
$$\begin{array}{ccc} E_1 & & E_2 \\ G \searrow & & \swarrow G \\ & S & \end{array} \qquad \begin{aligned} &\mathrm{Iso}(E_1, E_2)(S') \\ &= \mathrm{Iso}((E_1)_{S'}, (E_2)_{S'}) \end{aligned}$$

If $E_i = S \times G \Rightarrow \mathrm{Iso}(E_1, E_2) = G \times S$

idea: sheaf can be glued in étale topology

affine = ~~sh~~ sheaf of algebra

Fact: $X/\mathbb{C}$ projective

G alg. group of finite type $/\mathbb{C}$

$\mathrm{Bun}_G(X)$ is algebraic

$\mathrm{Bun}_{GL_r}(X) = \mathrm{Bun}_r(X)$

Ex. Fact: $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ F schematic

$\mathcal{Y}$ algebraic $\Rightarrow$ $\mathcal{X}$ algebraic

$$\begin{array}{c} \ell \\ \downarrow \\ X \end{array} \text{ line bundle}$$

$$\mathrm{Higgs}_r(X, \ell)(S) = \left\{ \begin{array}{c} E \\ \downarrow \\ X \times S \end{array}, \quad \begin{array}{l} \varphi : E \to E \otimes p^* \ell \\ \\ p : X \times S \to X \end{array} \right\}$$

$\mathrm{Higgs}_r(X) \xrightarrow{F} \mathrm{Bun}_r(X)$

Ex. F is schematic $\Rightarrow \mathrm{Higgs}_r(X)$ is algebraic

$\mathrm{Coh}(\mathcal{X})$ $\forall \alpha : \mathcal{X}(S)$ $S \xrightarrow{\alpha} \mathcal{X}$

· should be given F_S on S

- $S' \xrightarrow{f} S$ an iso $F_{S'} \xrightarrow{\varphi_f} f^* F_S$
- + compatibilities

Ex.

$$* = BG = */G$$

$$G \times * \rightarrow * \rightarrow */G$$

$F_{S \to *}$ — a vector space

Thm. $\mathrm{Coh}(BG) = \mathrm{Rep}^f(G)$

$\mathrm{Coh}(X/G) = \mathrm{Coh}^G(X)$

(Bundles with) connections (on curves)

① Systems of ODE

② Moduli stacks

$E \xrightarrow{\pi} X \leftarrow$ a scheme $/\mathbb{C}$ or a complex mfld

$E \rightsquigarrow \underline{E}(\mathcal{U}) = \{ s : \mathcal{U} \to E,\ \pi \circ s = Id \}$

$\underline{E}$ a locally free sheaf of $\mathcal{O}_X$-modules

Def. A connection on $\underline{E}$ is a $\mathbb{C}$-linear morphism

$$\underline{E} \xrightarrow{\nabla} \underline{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

$\leftarrow$ (general coherent sheaves does not have connection)

$\forall\, \mathcal{U} \subset X,\ s \in \underline{E}(\mathcal{U})\ \ f \in \mathcal{O}_X(\mathcal{U})$

$$\nabla(fs) = f \nabla s + s \otimes df$$

Warning: ∇ is not $\mathcal{O}_X$-linear

$v \in \Theta_X(\mathcal{U}) \qquad \Omega^1_X = \Theta_X^\vee$

$$\nabla_v s = \langle v, \nabla s \rangle \in \underline{E}(\mathcal{U})$$

$$\Theta_X \otimes \underline{E} \otimes_{\mathcal{O}_X} \Omega^1_X \to \mathcal{O}_X \otimes_{\mathcal{O}_X} \underline{E} \to \underline{E}$$

$\Theta_X \times \underline{E} \to \underline{E}$

$\Theta_X \otimes_{\mathbb{C}} \underline{E} \to \underline{E}$

$\not\downarrow \leftarrow$ Ex.

$\Theta_X \otimes_{\mathcal{O}_X} \underline{E} \to \underline{E}$

$E = \mathcal{O}_X \quad (X \times \mathbb{C})$

$d : \mathcal{O}_X \to \Omega_X = \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X$

$\alpha \in \Gamma(X, \Omega_X^1)$

Lemma. $d+\alpha$ is a connection, any connection is of this form.

Pf. (1) $\cancel{\nabla f = df + f\alpha}$

$\nabla g = dg + g\alpha$

$\nabla(fg) = d(fg) + fg\alpha$

$= \underbrace{\cancel{df \cdot g}}_{g \cdot df} + f \cdot dg + fg\alpha$

$= g \cdot df + f \cdot \nabla g$

(2) $1 \in \Gamma(X, \mathcal{O}_X)$

$\nabla \cdot 1 =: \alpha \in \Gamma(X, \Omega_X^1)$

$\nabla(g) = dg + g\alpha$ $\square$

$E = \mathcal{O}_X^{\oplus r}$ $\quad \in \mathcal{O}_X^{\oplus r} \quad \in \mathcal{O}_X^{\oplus r} \otimes \Omega_X^1 = \Omega_X^{\oplus r}$

$$d \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} = \begin{pmatrix} ds_1 \\ \vdots \\ ds_r \end{pmatrix}$$

$A \in \mathrm{Mat}_{r\times r}(\Gamma(X, \Omega_X)) \ni A_i^j$

$D = d - A$

$$D\begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} = \begin{pmatrix} ds_1 \\ \vdots \\ ds_r \end{pmatrix} - A \begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix}$$

$$\begin{tikzcd} E' \arrow[r, "\sim"] \arrow[d, "s'"'] & E \arrow[dl, "s"] \\ X & \end{tikzcd} \qquad \begin{tikzcd} D_s \in E \otimes \Omega_X \arrow[d, "\wr"] \\ E' \otimes \Omega_X \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_X^{\oplus r} \arrow[r, "\varphi \in GL_r(X)"] \arrow[dr] & \mathcal{O}_X^{\oplus r} \arrow[d] & d - A \arrow[l] \\ & X & \end{tikzcd}$$

$(d - A)(\varphi s) = d\varphi . s + \varphi \cdot ds - A\varphi s$

$$D'(s) = ds - (\underbrace{\varphi^{-1} A \varphi - \varphi^{-1} d\varphi}_{\text{gauge transf}})$$

$E \to X \qquad X = \bigcup_\alpha U_\alpha$ open cover

$E|_{U_\alpha} \xleftarrow{s_\alpha} \mathcal{O}_{U_\alpha}^{\oplus r}$

$D_\alpha = d - A_\alpha \qquad A_\alpha \in \mathrm{Mat}_{r\times r}(\Omega_X(U_\alpha))$

$$\begin{tikzcd}
E|_{U_\alpha \cap U_\beta} & \mathcal{O}^{\oplus r}_{U_\alpha \cap U_\beta} \arrow[l, "s_\alpha"'] \\
& \mathcal{O}^{\oplus r}_{U_\alpha \cap U_\beta} \arrow[u, "\varphi_{\alpha\beta}"'] \arrow[ul, "s_\beta"]
\end{tikzcd}$$

$$A_\beta = \varphi_{\alpha\beta}^{-1} A_\alpha \varphi_{\alpha\beta} - \varphi_{\alpha\beta}^{-1} d\varphi_{\alpha\beta}$$

X — smooth proj. cure/k, $\bar{k} = k$

$E \xrightarrow{\nabla} X$

Thm. (Weil) E has a connection $\Longleftrightarrow$

$\forall E = E' \oplus E''$, degree $E' = 0$ (in general, chern class $= 0$)

"Pf". $r = 1$

E l.b.

$U = X - \{x_1, \dots, x_n\}$

$E|_U \approx \mathcal{O}_U$

$E_{U_i} \approx \mathcal{O}_{U_i}$ $\quad x_i \in U_i \subset X$

$A \in \Omega_X(U)$

$A_i \in \Omega_X(U_i)$

$A_i = A - \frac{d\varphi_i}{\varphi_i}$ $\quad \varphi_i \in \mathcal{O}_X(U_i \cap U_j)$

$\uparrow$

1x1 matrix is abelian

$\deg E = \sum_i \operatorname{ord}_{x_i} \varphi_i$

$\sum_i \operatorname{res}_{x_i} A = 0$

$\operatorname{res} \frac{d\varphi_i}{\varphi_i} = \operatorname{ord}_{x_i} \varphi_i = -\operatorname{res}_{x_i} A$

$(E, \nabla) \quad r > 1$

$\deg E = \deg \Lambda^r E = 0$

$$\begin{array}{c} E' \hookrightarrow E' \oplus E'' \xrightarrow{\nabla} E' \otimes \Omega_X \oplus E'' \otimes \Omega_X \\ \downarrow \\ E'' \otimes \Omega_X \end{array}$$

smooth connected

$X/\mathbb{C}$ a curve (E, ∇) ($\mathbb{C}$-analytic)

$E^\nabla(U) = \{ s \in E(U), \nabla s = 0 \}$

$E = \mathcal{O}_X^{\oplus r} \Leftarrow$ choose a coordinate z on X

$\nabla = d - A$

$ds = As \qquad A = B(z) \cdot dz$

$\boxed{\frac{ds}{dz} = B(z) \cdot s}$ DE

Claim. E^∇ is locally isomorphic to $\mathbb{C}_X^{\oplus r}$

not a sheaf of $\mathcal{O}_X$-module

Pf. May assume $X = \{|z| < 1\} \subset \mathbb{C}$

May assume $E = \mathcal{O}_X^{\oplus r}$

$\forall\ (c_1, \cdots, c_r) \in \mathbb{C}^r$

$\exists!\ s_1, \ldots, s_r : \ \frac{ds}{dz} = B(z) \cdot s$

$s_i(0) = c_i$ [Main Thm of complex ODE's]

$\square$

$\mathrm{Conn}(X)$

$E_1 \xrightarrow{\alpha} E_2$

$\nabla_2 \alpha(s) = (\alpha \otimes \mathrm{Id}_{\Omega_X}) \nabla_1(s)$

$\mathrm{Conn}(X) \xrightarrow[RH]{\text{Functor}} \mathrm{LocSys}(X)$ $=$ (sheaves locally isomorphic to $\mathbb{C}_X^{\oplus r}$)

Thm. (RH - correspondence)

an equivalence of categories

Rem. GAGA : If X is projective

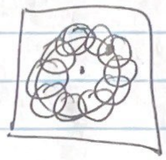
$\mathrm{Conn}_{an}(X) \simeq \mathrm{Conn}_{alg.}(X)$

If not, can have

$\nabla = d + e^z dz$

Prop. Pick $x \in X$. Then

$\mathrm{LocSys}(X) \simeq \mathrm{Rep}^{f.d.}(\pi_1(X^{\circ}, x))$

$\mathbb{C}^*$  monodromy

X - variety / $\mathbb{C}$

$E \to X$ rank r v.b.

$\nabla : E \to E \otimes \Omega^1_X$

$\mathbb{C}$-linear, Leibnitz rule

$E^\nabla \subseteq E$ $\quad E^\nabla(U) = \{ s \in E(U),\ \nabla|s = 0 \}$

$\dim X = 1$ $\quad E^\nabla$ is a local system

(locally iso $\sim \mathbb{C}_X^{\oplus r}$)

$\dim X = 2$ $\quad t_1, t_2$ local coordinates on X

PDE $\quad \begin{cases} \dfrac{\partial s}{\partial t_1} = A(t_1, t_2) s \\ \dfrac{\partial s}{\partial t_2} = B(t_1, t_2) s \end{cases}$

$$\frac{\partial^2 s}{\partial t_1 \partial t_2} = \frac{\partial B}{\partial t_1} s + A \frac{\partial s}{\partial t_2} = \frac{\partial B}{\partial t_1} s + ABs$$

$$\frac{\partial B}{\partial t_1} + AB = \frac{\partial A}{\partial t_2} + BA$$

$$\frac{\partial B}{\partial t_1} - \frac{\partial A}{\partial t_2} + [A, B] = 0$$ flat connection

$$E \xrightarrow{\nabla} E \otimes \Omega^1_X \xrightarrow{\nabla} E \otimes \Omega^2_X \to \cdots \xrightarrow{\nabla} E \otimes \Omega^{\dim X}_X$$

$\nabla(s\otimes w) = \underbrace{Ds\wedge w} + s\otimes dw \quad (\, \nabla(fs\otimes w) = \nabla(f\otimes sw))$

$$Ds = \sum s_i \otimes w_i$$

$$Ds\wedge w := \sum s_i \otimes (w_i \wedge w)$$

Ex. (i) ∇ is flat $\iff \nabla^2 : E \to E \otimes \Lambda^2 \Omega_X \quad \nabla^2 = 0$

(ii) ∇^2 is $\mathcal{O}_X$-linear

$\mathrm{FlatConn}(X) \xrightarrow{\sim} \mathrm{LocSys}(X)$

(Riemann - Hilbert)

$\mathrm{LocSys}(X) \xrightarrow{\otimes \mathcal{O}_X} \mathrm{FlatConn}(X)$

$\mathcal{L}$ local system on X

$\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$ locally iso to $\mathcal{O}_X^{\oplus r}$

sheaf of sections of a vector bundle

$\nabla(s\otimes f) = s\otimes df$

X - compact $\mathbb{C}$ - mfld (algebraic)

(GAGA, Cartan) $\quad \mathcal{V}$ category of v.b.

$$\mathcal{V}_{an}(X) \xrightarrow{\sim} \mathcal{V}_{alg}(X)$$

X

$\mathcal{D}_X\text{-mod} = \{\text{Quasi-Coh}(X) \text{ with flat connections}\}$

$\uparrow$

$\text{Coh}(X)$ with flat connections

$\Downarrow$

v.b.

alg. not isomorphic

$\text{LocSyms}(X) = \text{Rep}(\Pi_1(X)) \simeq \text{Rep}(\Pi_1(X, x))$

$\text{Obj}(\Pi_1(X)) = X$

$\text{Mor}_{\Pi_1(X)}(x, y) = \{\gamma : x \to y\} / \text{homotopy}$

If X is connected, $x \in X$

$\Pi_1(X) \simeq B\,\Pi_1(X, x)$

$\mathcal{F}$ local system

$\gamma : [0, 1] \to X$

$\gamma^* \mathcal{F} \simeq \mathbb{C}^{\oplus r}_{[0,1]}$

$$\begin{array}{ccc} \gamma^*\mathcal{F}|_0 & \simeq & \gamma^*\mathcal{F}|_1 \\ \| & & \| \\ \mathcal{F}_x & \to & \mathcal{F}_y \end{array}$$

X - compact alg. curve

$\text{Conn}_r(X)$ - algebraic stack

$\mathrm{Conn}_r(X) \to \mathrm{Bun}_r(X)$

$(E, D) \mapsto E$ is schematic

$$\begin{array}{ccc} \mathrm{Conn}_r(X) & \to & \mathrm{Bun}_r(X) \\ \uparrow & & \uparrow \; E \to X \\ \mathrm{Conn}(X, E) & \to & \mathrm{Spec}\,\mathbb{C} \end{array}$$

Either empty or $\sim H^0(X, \mathrm{End}(E) \otimes \Omega_X)$

If not empty:

$D, D_0 : E \to E \otimes \Omega^1_X$

$D - D_0 : E \to E \otimes \Omega^1_X$

$D - D_0 \in \mathrm{Hom}_{\mathcal{O}_X}(E, E \otimes \Omega^1_X)$

$= H^0(X, E^\vee \otimes E \otimes \Omega^1_X)$

Higgs fields

$\{a_i, b_i, i = 1, \dots, g, \prod a_i b_i a_i^{-1} b_i^{-1} = 1\}$

$\mathrm{Rep}(\pi_1(X, x)) \simeq \{A_i, B_i \in \mathrm{Mat}_{r \times r}, \prod A_i B_i A_i^{-1} B_i^{-1} = 1\} / GL(r)$

$\quad \parallel$

$\mathrm{Spec}\, R^{GL(r)}$

affine algebraic variety

we trivialize the fiber

Ex. $E \overset{\text{analytic}}{\simeq} \mathbb{C}^*/\mathbb{Z}$ $\quad \mathrm{Conn}_1(E)$

$\qquad\qquad\qquad\qquad \downarrow$

Weil's theorem $\mathrm{Bun}_{1,0}(E) \simeq E$

$\mathrm{LocSys}_1(E) \simeq \mathbb{C}^* \times \mathbb{C}^* \leftarrow$ lots of global functions

$\Gamma(\mathrm{Conn}_1(E), \mathcal{O}) = \mathbb{C} \leftarrow$ only constant

alg.

$Y = Y_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$ category of smooth alg.

$\downarrow$ smooth

curve $\rightarrow x \in X = X_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$

$(\mathcal{L}_Y)_x = H^d(Y_x, \mathbb{C}) \supset H^d_x(Y, \mathbb{Z}) \leftarrow$ monodromy

local system

$\cup$

$\mathcal{L}_{Y,\mathbb{Z}} \rightsquigarrow (E_Y, \nabla_Y)$ De Rham

(Gauss-Manin)

Ex. $X = \mathbb{C}^*$ $r = 1$

$\nabla = d + \frac{a}{z} dz$, $a \in \mathbb{C}^*$

$M = e^{2\pi i a}$

$\bar{\mathbb{Q}}$: $a, e^{2\pi i a} \in \bar{\mathbb{Q}} \Leftrightarrow a \in \mathbb{Q}$

X compact algebraic curve

$(E, \nabla) \subset \mathrm{Conn}_r(X)$

irreducible Geometric Langlands

$(E, \nabla) \rightsquigarrow \mathcal{H}_{E,\nabla} \in \mathcal{D}\text{-mod}(\mathrm{Bun}_r(X))$

generalize to a family of connections

$$\mathrm{Coh}(\mathrm{Conn}_r(X)) \simeq \mathcal{D}\text{-mod}(\mathrm{Bun}_r(X))$$

X not compact?

$$X = \mathbb{P}^1 \setminus \{x_1, \dots, x_k\}$$

$\mathrm{Conn}(X)$ and $\mathrm{Bun}(X)$ are not alg. stacks

$\{E \xrightarrow{r} \mathbb{P}^1, E \to E \otimes \Omega_{\mathbb{P}^1}\}$

$\nwarrow \mathcal{O}(-2)$ few sections

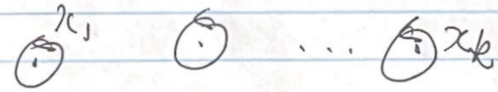
$$\{E \xrightarrow{r} \mathbb{P}^1, E \xrightarrow{\nabla} E \otimes \Omega_{\mathbb{P}^1}(x_1 + \cdots + x_k)\}$$

$$= \mathrm{Conn}_r(\mathbb{P}^1 - \{x_1, \dots, x_k\})$$

$E \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$

$$\nabla = d + \sum \frac{A_i}{z - x_i} dz \qquad \sum A_i = 0$$

$A_i \in \mathrm{Mat}_{r \times r}(\mathbb{C})$ & Fix conjugacy classes



$$\pi_1(\mathbb{P}^1 \setminus \{x_1, \dots, x_k\}) = \{a_1 \cdots a_k = 1\}$$

$$= F_{k-1}$$

$$\mathrm{LocSys}(\mathbb{P}^1 \setminus \{x_1, \dots, x_k\}) = \{B_1 \cdots B_k = 1\}$$

$\uparrow$ Fix conjugacy classes

Ex. $\mathbb{P}^1 - \{0, \infty\}$

$$D = d + \frac{A}{z} dz \qquad A \text{ constant}$$

$$M = e^{2\pi i A} \qquad RH: \ A \mapsto e^{2\pi i A}$$

$$gl_{r\times r} \twoheadrightarrow Gl_{r\times r}$$

$$\not\hookleftarrow$$

~~Con~~ $\mathrm{Coh}(\mathrm{Conn}(X - \{x_1, \cdots, x_n\}))$

tamely ramified

~~Ex~~. $\mathbb{P}^1$, $2(0)+(\infty)$ irregular

$$\left\{d + \left(\frac{A}{z^2} + \frac{B}{z}\right) dz\right\} = \mathrm{Conn} \xrightarrow{\sim} \mathrm{LocSys}$$

stokes data

$X \supset D \qquad D = \sum n_i x_i$, $n_i > 0$

(wild, irregular)

$\mathcal{D}$- modules

I. Bernstein V. Ginzburg (notes)

Hotta, Takeuchi, Tanisaki

X - smooth complex variety (complex mfld)

$\mathcal{D}_X$ - sheaf of linear diff. op.

$Mod(\mathcal{D}_X)$ left modules

Ex. $X = \mathbb{A}^1_{\mathbb{C}}$

$$\left\{ \sum_{i=0}^{n} a_i(z) \frac{\partial^r}{\partial z^r} \right\} = \mathcal{D}_{\mathbb{A}^1} = \langle z, \tfrac{\partial}{\partial z} \mid [z, \tfrac{\partial}{\partial z}] = 1 \rangle$$

$$\circlearrowright$$

$$\mathbb{C}[z]$$

$$\underset{\mathcal{D}_{\mathbb{A}^1}}{\underset{\Downarrow}{\left(z \frac{\partial}{\partial z}\right)}} \cdot \underset{\mathbb{C}[z]}{\underset{\Downarrow}{z^3}} = z \cdot 3z^2 = 3z^3$$

$\mathcal{D}_{\mathbb{A}^1}$ - modules

- $\mathbb{C}[z] \leftarrow \frac{\partial}{\partial z} \cdot 1 = 0 \quad z \cdot 1 = z$
- $\mathbb{C}[z] \quad \frac{\partial}{\partial z} \cdot 1 = f \in \mathbb{C}[z]$

$$\frac{\partial}{\partial z} \cdot (z) = \frac{\partial}{\partial z} \cdot (z \cdot 1)$$

$$= z \frac{\partial}{\partial z} \cdot 1 + 1 \cdot 1$$

$$= zf + 1$$

(E, ∇) with $\nabla^2 = 0 \rightsquigarrow \mathcal{D}$-module on X

δ : $\delta \in M_\delta$ $\quad z \cdot \delta = 0$

$$\begin{gathered} \delta \in M_\delta \\ \| \\ \mathbb{C}\delta \oplus \mathbb{C}\delta' \oplus \mathbb{C}\delta'' \oplus \cdots \end{gathered} \qquad \frac{d}{dz}\delta^{(k)} = \delta^{(k+1)}$$

$I \subset R \quad I, R/IR \in \mathrm{Mod}(R)$

$\uparrow$

left ideal

$P \in \mathcal{D}_{\mathbb{A}^1}$

$\mathcal{D}/\mathcal{D}P \in \mathrm{ModQ}(\mathcal{D}_{\mathbb{A}^1})$

Ex. $\mathcal{D}/\mathcal{D}\frac{d}{dz} \simeq \mathbb{C}[z]$

$\mathcal{D}/\mathcal{D}z \simeq M_\delta$

Now $\mathcal{D} = \mathcal{D}_{\mathbb{A}^1}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}P, \mathbb{C}[z]) = \{f \in \mathbb{C}[z] : Pf = 0\}$$

$$\mathrm{Hom}_R(R/aR, M) = \{m \in M : am = 0\}$$

$$\mathcal{D}_{\mathbb{A}^n} = \langle z_1, \dots, z_n, \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} ; [z_i, z_j] = 0, \ [\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}] = 0, \ [z_i, \frac{\partial}{\partial z_j}] = \delta_{ij} \rangle$$

X - smooth

$\mathcal{D}_X \subset \cancel{\text{Diff}}\,\overset{\text{End}}{}_{\mathbb{C}}(\mathcal{O}_X)$ $\quad (\text{End}_{\mathbb{C}}\mathcal{O}_X)(U) = \{\mathcal{O}_X(U) \to \mathcal{O}_X(U),$

$\mathbb{C}$-linear$\}$

$\mathcal{O}_X \in \text{End}_{\mathbb{C}}(\mathcal{O}_X)$

$f \in \mathcal{O}_X(U) \rightsquigarrow (g \mapsto fg)$

$\Theta_X \subset \text{End}_{\mathbb{C}}(\mathcal{O}_X)$ $\quad$ vector fields

$v \in \Theta_X(U) \rightsquigarrow (f \mapsto vf)$

$\mathcal{D}_X$ is the subsheaf of $\text{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and Θ_X.

$\mathcal{D}_X = \langle \mathcal{O}_X \oplus \Theta_X \mid vf - fv = v(f),$

sheaf of $\mathbb{C}$-algebras $\qquad v_1v_2 - v_2v_1 = [v_1, v_2] \rangle$

$(vf - fv)g = v(fg) - f(vg)$

$= v(f)\cdot g$

Quot $\quad \bigoplus_0^\infty (\mathcal{O}_X \oplus \Theta_X)^{\otimes n}$

A $\mathcal{D}$-module on X: a sheaf F of $\mathcal{O}_X$-modules

$\Theta_X \otimes_{\mathbb{C}} F \xrightarrow{a} F$

$$a(fv\otimes m) = f\cdot a(v\otimes m) - v(f)\cdot m$$

$$a(v_1\otimes a(v_2\otimes m)) - a(v_2\otimes a(v_1\otimes m))$$
$$= a([v_1,v_2]\otimes m)$$

Let $\mathcal{F}$ be a $\mathcal{D}_X$-module

View $\mathcal{F}$ as an $\mathcal{O}_X$-module

$\mathcal{F} \xrightarrow{\nabla} \mathcal{F}\otimes\Omega_X$ dual to a

locally TX has a frame $v_1, \cdots, v_n$

dual frame of T^*X: $\alpha_1, \cdots, \alpha_n$

$$\nabla(m) = \sum a(v_i\otimes m)\otimes\alpha_i$$

Prop. (1) ∇ factors through a connection

$$\mathcal{F} \to \mathcal{F}\otimes_{\mathcal{O}_X}\Omega_X$$

(2) $\nabla^2 = 0$

(3) If $\nabla: \mathcal{F}\to\mathcal{F}\otimes_{\mathcal{O}_X}\Omega_X$ is a flat connection, then it comes from a unique $\mathcal{D}_X$-module via the construction

$$\mathcal{D}_X\text{-mod}^{coh} \subset \underset{\substack{\uparrow \\ \text{quasi-coh as } \mathcal{O}_X\text{-modules}}}{\mathcal{D}_X\text{-mod}^{qc}} \subset \mathcal{D}_X\text{-mod}$$

Theorem. Assume that $\mathcal{F}$ is a $\mathcal{D}_X$-module coherent as an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is locally free. (i.e. vector bundle)

M_δ – not coherent as an $\mathcal{O}_{\mathbb{A}^1}$-module

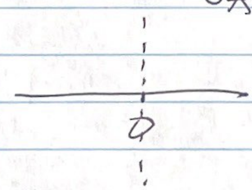
$f : Y \to X$

$f^* : \mathcal{D}\text{-mod}(X) \to \mathcal{D}\text{-mod}(Y)$

$f_* : \mathcal{D}\text{-mod}(Y) \to \mathcal{D}\text{-mod}(X)$

Ex.
$$\begin{tikzcd} 0 \arrow[r, hook, "i"] & \mathbb{A}^1 \\ \mathbb{C} \arrow[u] & \end{tikzcd}$$

$i_* \mathbb{C} = M_\delta$



$$\mathbb{A}^1 - \{0\} \xhookrightarrow{j} \mathbb{A}^1 \xhookleftarrow{i} 0$$

$\mathbb{C}[z, \frac{1}{z}]$

$j_* \mathbb{C}[z, \frac{1}{z}] = \mathbb{C}[z, \frac{1}{z}]$

$0 \to \mathbb{C}[z] \hookrightarrow \mathbb{C}[z, \frac{1}{z}] \to M_\delta \to 0$

$\frac{1}{z} \mapsto \delta$

$z \cdot \frac{1}{z} \mapsto 0$

$$\frac{(-1)^n n!}{z^{n+1}} \mapsto \delta^{(n)}$$

$$0 \to F \to j_* j^* F \to i_* i^* F \to 0 \qquad F = \mathbb{C}[z]$$

Beilinson - Bernstein localization

$$\mathcal{D}\text{-mod}^{qc}(\mathbb{P}^1) \xrightarrow{\Gamma} \mathrm{Rep}(pgl(2))$$

$$PGL(2) \circlearrowright \mathbb{P}^1$$

$$pgl(2) \hookrightarrow \mathrm{Vect}(\mathbb{P}^1) \subset \mathcal{D}(\mathbb{P}^1)$$

G - reductive $\quad P \subset G$ parabolic

$$\mathcal{D}\text{-mod}(G/P) \xrightarrow{\Gamma} \mathrm{Lie}(G)\text{-mod}$$

functoriality $\quad f_* / f^*$

Riemann - Hilbert

$\mathcal{D}_X$ - modules

$$\mathcal{D}_X^{l}\text{-mod} \simeq \mathcal{D}_X^{r}\text{-mod}$$

$$M \rightsquigarrow M \otimes_{\mathcal{O}_X} \omega_X$$

$$\omega_X = \Lambda^{\dim X} \Omega_X$$

Ex. $\quad X \quad \mathcal{O}_X \in \mathcal{D}_X^{l}\text{-mod}$

$v \cdot f = vf$ $\quad X$ curve of genus $g \neq 0$

$\mathcal{O}_X$ - has no structure of ~~righ~~ right $\mathcal{D}$-module

ω_X^{-1} - no structure of left $\mathcal{D}$-module

$f: \cancel{Q} Y \to \cancel{\varphi} X$

$$Sh(\cancel{X} Y, \mathbb{C}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} Sh(\cancel{\varphi} X, \mathbb{C})$$

$$\mathcal{O}_{\cancel{X} Y}\text{-mod} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{O}_{\cancel{\varphi} X}\text{-mod} \qquad f^*(M) = \cancel{f^{-1}\mathcal{D}} \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}M$$

$$\mathcal{D}_Y\text{-mod} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{D}_X\text{-mod}$$

$M \in \mathcal{D}_X$ - mod

$f^*M \in \mathcal{O}_Y$ - mod

$v \in \Theta_Y(U)$ vector field

$$\underset{\mathcal{O}_Y}{\underset{\cap}{\varphi}} \otimes \underset{M(f(U))}{\underset{\cap}{m}} \in f^*M$$

$v(\varphi \otimes m) = v\varphi \otimes m + \varphi \otimes \underline{vm}$

$= \sum \psi_i \otimes v_i m$

$v_i m \in M(f(U)) = f^{-1}(M(U))$

$p \in Y \qquad \cancel{(vm)_p = ...}$

$$TY \xrightarrow{df} f^* TX$$

$(df)(v) = \sum \psi_i \otimes u_i \qquad u_i \in \mathcal{O}_X(f(U))$

$\psi_i \in \mathcal{O}_Y(U)$

X, Y smooth quasi-proj.

lemma. Let $f: X \to Y$ be quasi-proj.

$f = i \circ j \circ p \qquad i$ closed embedding

j open embedding

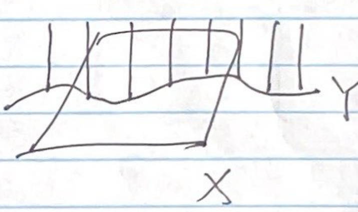
p smooth projective

$X \hookrightarrow Y \times \mathbb{P}^n \xrightarrow[p]{} Y$

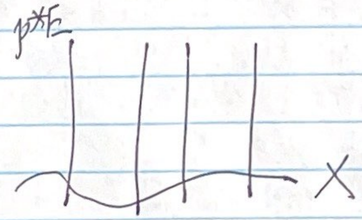
$j: Y \xhookrightarrow[\text{open}]{} X \qquad i: Y \xhookrightarrow[\text{closed}]{} X$

$j^* M = M|_Y \qquad i^* M = M|_Y$

$\cancel{i^* M}$



$Y \xrightarrow{\text{smooth}} X$



$V \otimes \mathcal{O}_Y$ vector bundle

$Y \to pt = X$

V vector space

$$f^*\mathcal{D}_X = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_X$$

$$\parallel$$

$\mathcal{D}_{Y\to X}$ — $\mathcal{D}_Y$ - $f^{-1}\mathcal{D}_X$ - bimodule

$M \in \mathcal{D}_Y$ - mod^r

$$M \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X} \in f^{-1}\mathcal{D}_X\text{- mod}^r$$

$$f_\star M = f_*(M \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}) \in \mathcal{D}_X\text{- mod}^r$$

$\begin{cases} \text{left } \mathcal{D}\text{-module function} \\ \text{right } \mathcal{D}\text{-module distribution} \end{cases}$

$$\omega_X \otimes \mathcal{D}_{Y\to X} \otimes f^{-1}\omega_X^{-1} =: \mathcal{D}_{X\leftarrow Y}$$

— $f^{-1}\mathcal{D}_X$ — $\mathcal{D}_Y$ - module

$$f_\star M = Rf_*(\mathcal{D}_{X\leftarrow Y} \overset{L}{\otimes}_{\mathcal{D}_Y} M)$$

Ex.

$f = j : Y \hookrightarrow X$ $\mathcal{D}_{Y\to X} = \mathcal{D}_Y$

$$j_\star M = j_* M$$

$$Y = \mathbb{C}^* \overset{j}{\hookrightarrow} \mathbb{C} = X$$

$\leftarrow \mathbb{C}[z, z^{-1}]$

$$M = (\mathcal{O}_{\mathbb{C}^*}, d + \mu \frac{dz}{z})$$

$$j_* M = (\mathbb{C}[z, z^{-1}],\ d + \mu \tfrac{dz}{z}) \in \mathcal{D}\text{-mod}(\mathbb{C})$$

Exercise. When $M_\mu \simeq M_{\mu'}$?

$\mu = 0$ $\quad j_{!*}(M_0) = (\mathbb{C}[z], d)$

bundle with ~~sign~~ singular connection / curve

$\Downarrow$

$\mathcal{D}_X / \mathcal{D}_X I_Y$ $\qquad$ $\mathcal{D}$ - module

$i: \ Y \hookrightarrow X \rightarrow$ locally $\mathbb{A}^m \hookrightarrow \mathbb{A}^n$

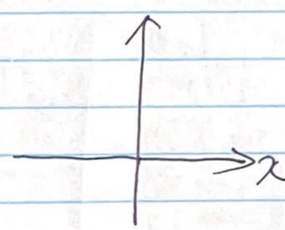
$M \in \mathcal{D}\text{-mod}(Y)$

$i_* M$ - supported on Y set-theoretically

$\uparrow$

not coherent as $\mathcal{O}_X$ - module

Ex.



$\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$

$$\begin{aligned}
\mathcal{D}_{\mathbb{A}^2}|_{\mathbb{A}^1} &= \{ \textstyle\sum f_{i,j}(x,y)\, \partial_x^i \partial_y^j \} \\
&= \{ \textstyle\sum f_{i,j}(x)\, \partial_x^i \partial_y^j \} \\
&= \bigoplus \mathcal{D}_{\mathbb{A}^1} \partial_y^j
\end{aligned}$$

$$\mathcal{D}_{Y \to X} \otimes_{\mathcal{D}_Y} M = \Big(\bigoplus \mathcal{D}_{\mathbb{A}^1} \partial_y^j\Big) \otimes_{\mathcal{D}_{\mathbb{A}^1}} M = \bigoplus M \partial_y^j$$

$M = \mathcal{O}_Y \sim \mathbb{C}[x]$

$1 \otimes i_* M = \bigoplus \mathbb{C} x^i \frac{\partial}{\partial y^j}$ $\leftarrow$ δ-functions in the direction of y

$x \cdot 1 = x, \frac{\partial}{\partial x} \cdot 1 = 0 \qquad y^{j+1} \frac{\partial}{\partial y^j} = 0$

$y \cdot 1 = 0, \frac{\partial}{\partial y} \cdot 1 = \frac{\partial}{\partial y} \qquad y^j \frac{\partial}{\partial y^j} = \text{const} \neq 0$

$\mathcal{D}_{\mathbb{A}^2} / (x, \frac{\partial}{\partial y})$

Exercise. $\cdot \xrightarrow[i]{} \mathbb{A}^1$

$i_* \mathbb{C} \in \mathcal{D}_{\mathbb{A}^1}\text{-mod}$

Kashiwara's Thm

$Y \subset X$ is a closed embedding

(i) i_* is exact, fully faithful

(ii) The essential image of i_* consists of $\mathcal{D}_X$-modules supported on Y <u>set-theoretically</u>

Remarks ① $\mathcal{F} \in \mathrm{Coh}(X)$

$\mathrm{supp\,set}(\mathcal{F}) \subset X$

Not necessarily $\mathcal{F} \simeq i_* \mathcal{G}$

② $\mathcal{D}_Y(X)\text{-mod} = \{\mathcal{F} \in \mathcal{D}_X\text{-mod}, \text{set supp}(\mathcal{F}) \subset Y\}$

$\mathcal{D}_Y(X)\text{-mod} \simeq \mathcal{D}(X)\text{-mod}$

^ makes sense if Y singular

Def. Y singular variety, q-proj.

embed $Y \underset{\text{closed}}{\hookrightarrow} X$ X smooth : $D(Y)\text{-mod} := \mathcal{D}_Y(X)\text{-mod}$

Recall. $f: Y \to X$ morphism of smooth varieties/$\mathbb{C}$

$f^{-1}: Sh_{\mathbb{C}}(Y) \leftrightarrows Sh_{\mathbb{C}}(X) : f_*$ derived category

$f^* = f^{-1} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y : QCoh(Y) \leftrightarrows QCoh(X) : f_*$

$f^* : Dmod(Y) \leftrightarrows Dmod(X) : f_\star$

$\mathcal{D}_{Y \to X} := f^* \mathcal{D}_X$ $\quad \mathcal{D}_Y - f^{-1}\mathcal{D}_X$-module

$\mathcal{D}_{X \leftarrow Y} := \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \to X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1}$ $\quad \omega_Y = \Lambda^{top}\Omega_Y$

$f^{-1}\mathcal{D}_X - \mathcal{D}_Y$-module

$M \in Dmod(Y)$

derived version $Rf_*(\mathcal{D}_{X \leftarrow Y} \overset{L}{\otimes}_{\mathcal{D}_Y} M)$ - $\mathcal{D}_X$-module

$f = p \circ i \circ j$ — p: smooth, i: closed embedding, j: open

Kashiwara $\quad i: Y \hookrightarrow X$ closed embedding

$i_\star : Dmod(Y) \to \mathcal{D}mod_Y(X) = \{M : M|_{X-Y} = 0\}$

$Y = \{0\} \hookrightarrow \mathbb{A}^1 = \mathbb{C} = X$

lemma. $M \in \mathcal{D}\text{-mod}(\mathbb{A}^1)$ $\quad M|_{\mathbb{A}^1 - \{0\}} = 0$

$\exists V: i_\star V \simeq M.$

Proof. M - $\mathbb{C}\langle x, \partial\rangle$ - module $\quad \partial x - x\partial = 1$

$\forall m \in M$

x acts locally nilpotent $\Rightarrow$ $x^n m = 0 \quad \forall n >> 0$

i.e. $M \otimes_{\mathbb{C}[x]} \mathbb{C}[x, x^{-1}] = 0$

$I := x\partial \qquad M_i := \{m : Im = im\}$ (will see $M = M_{-2} \oplus M_{-2} \oplus \cdots$)

$M_{i-1} \underset{\partial}{\overset{x}{\rightleftarrows}} M_i \qquad m \in M_{i-1} \quad I(xm) = x\partial x m$

$= x(x\partial + 1)m$

$= x(I+1)m$

$= x((i-1)+1)m$

$= i x m$

$I|_{M_i}$ iso if $i \neq 0$

$\Rightarrow x, \partial$ iso if $i \neq 0, -1$

$\partial x = I + 1$ iso if $i \neq -1$

$xm = 0 \Rightarrow m \in M_{-1}$

$x\partial m = (\partial x - 1)m = -m$

Claim. $x^k m = 0 \Rightarrow m \in M_{-1} \oplus \cdots \oplus M_{-k}$

$m = (\partial x - x\partial)m = \underset{\sim}{\partial x m} - x\partial m$

$xm \in M_{-1} \oplus \cdots \oplus M_{-k}$

$\partial x m \in M_{-2} \oplus \cdots \oplus M_{-k}$

$\cancel{x^{k-1} x \partial m = x^k \partial m = \partial x^k m - k x^{k-1} m}$

$\cancel{\quad \underset{0}{\shortparallel} \qquad \uparrow \text{ annihilated by } x^{k-1}}$

$\cancel{\Rightarrow x \partial m \in M_{-1} \oplus \cdots \oplus M_{1-k}}$

$m' = x \partial m$

$\cancel{x^{k-1} m' = x^k \partial m = -k x^{k-1} m}$

$$x m' = x^2 \partial m = \underbrace{\partial x^2 m}_{\in M_{-1} \oplus \cdots \oplus M_{1-k}} - 2 x m$$

$$x : M_{-2} \oplus \cdots \oplus M_{-k} \xrightarrow{\sim} M_{-1} \oplus \cdots \oplus M_{1-k}$$

$$\Rightarrow m' \in M_{-2} \oplus \cdots \oplus M_{-k}$$

Now $\quad M = V \oplus \partial V \oplus \partial^2 V \oplus \cdots = \mathbb{C}[\partial] \otimes_{\mathbb{C}} V \simeq i_* V$ $\blacksquare$

In general $\quad Y \subset X \quad$ locally hypersurface

$$Y \subset \cdots \subset X^2 \subset X' \subset X$$

$X' = \{f = 0\}$ $\qquad X' \xrightarrow{x} \qquad [\partial, x] = 1$

$(i_1 \circ i_2)_* = (i_1)_* \circ (i_2)_*$

$M \xrightarrow{p} pt$

$\mathcal{D}_{Y\to X} = \mathcal{O}_M$

$\mathcal{D}_{X\leftarrow Y} = \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y = \omega_Y$

$p_* M = Rp_* (M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \omega_Y) = Rp_{*}(M \otimes_{\mathcal{D}_Y} DR(\mathcal{D}_Y))$

$= Rp_{*}(DR(M))$

Proof. $M \in \mathcal{D}mod(Y)$

$DR(M) := M \xrightarrow{d_1} M \otimes \Omega_Y \xrightarrow{d_2} M \otimes \Lambda^2 \Omega_Y \xrightarrow{d_3} \cdots \to M \otimes \Lambda^n \Omega_Y$

$d^2 = 0$ because "connection is flat" $n = \dim Y$

$DR(\mathcal{D}_Y) = \cdots \to \mathcal{D}_Y \otimes \Lambda^{n-1}\Omega_Y \to \mathcal{D}_Y \otimes \Lambda^n \Omega_Y$

$\Lambda^n \Omega_Y = \omega_Y$

$\mathcal{D}_Y \to \mathcal{O}_Y$

$(f(x) + \sum f_i \frac{\partial}{\partial x_i} + \cdots) \mapsto f$

$DR(\mathcal{D}_Y) \underset{qiso}{\xrightarrow{\sim}} \omega_Y$

$\blacksquare$

Cor. $R^i p_* M = R^i p_* (DR(M)) = H^i(DR(M))$

if M affine

$R^i p_* \mathcal{O}_Y = R^i p_* (DR(Y)) \simeq H^i(Y, \mathbb{C})$

$Y \xrightarrow{p} X$ smooth

$DR_{Y/X}(M) = (M \to M \underset{\mathcal{O}_X}{\otimes} \Omega_{Y/X} \to M \otimes \Lambda^2 \Omega_{Y/X} \to \cdots)$

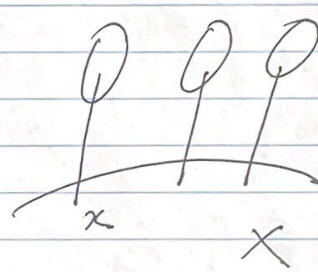
$Rp_* M = Rp_*(DR_{Y/X}(M))$ $\mathcal{D}_X$-module

$f: Y \to X$ locally trivial fibration projective

$M \sim (E, \nabla_E)$ $\mathrm{rk}\, E < \infty$

$R^i p_* \mathcal{O}_Y \sim (F, \nabla_F) \leftarrow$ Gauss-Manin connection

$F_x = H^i(Y_x, \mathbb{C})$ $Y_x := f^{-1}(x)$



locally/X

$Y \approx F \times X$

$Y_x \approx F$

$H^i(Y_x) \simeq H^i(Y_{x'})$

$x \to H^i(Y_x)$

locally constant sheaf of vector space